Consiglio Nazionale delle Ricerche
Istituto di Elettronica e di Ingegneria dell'Informazione e delle Telecomunicazioni

# A probabilistic scaling approach to chance constraints

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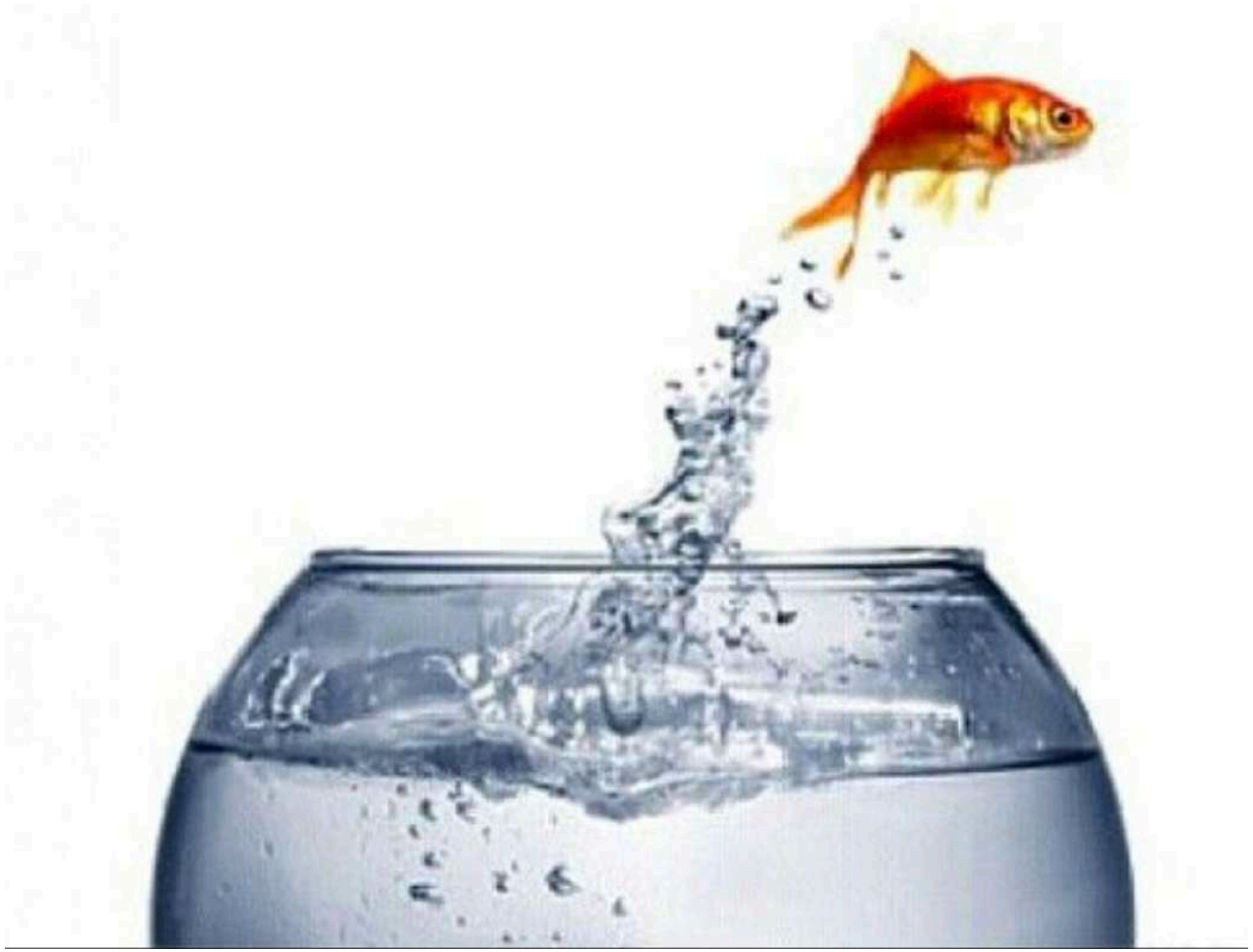
*Erice*

*May 2022*

# *piscem natare doces*



# Design under uncertainty

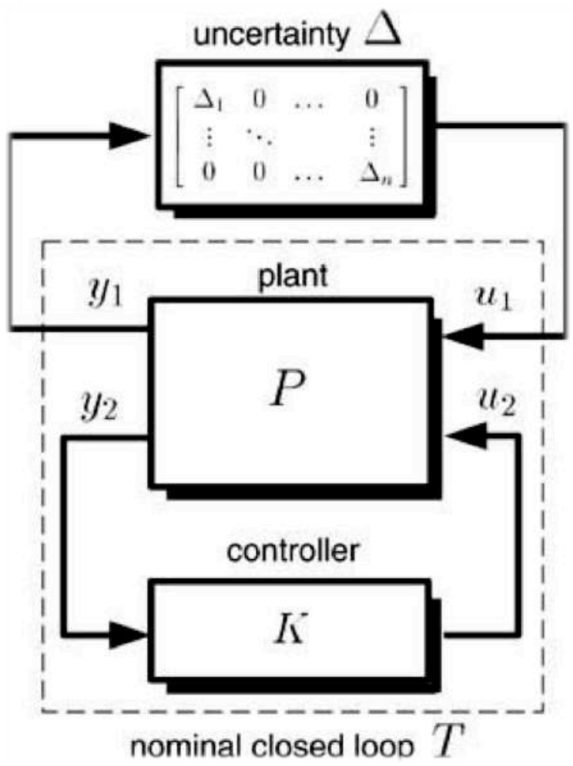
*"solum certum*
*nihil esse certi"*

**Plinius the old,**
Naturalis Historiæ, 77 d.C

- Modern optimization problems are characterized by an imperfect knowledge of the design environment
- Coping in an efficient way with uncertainty represents a key issue

# Uncertainty in control systems: Robust control

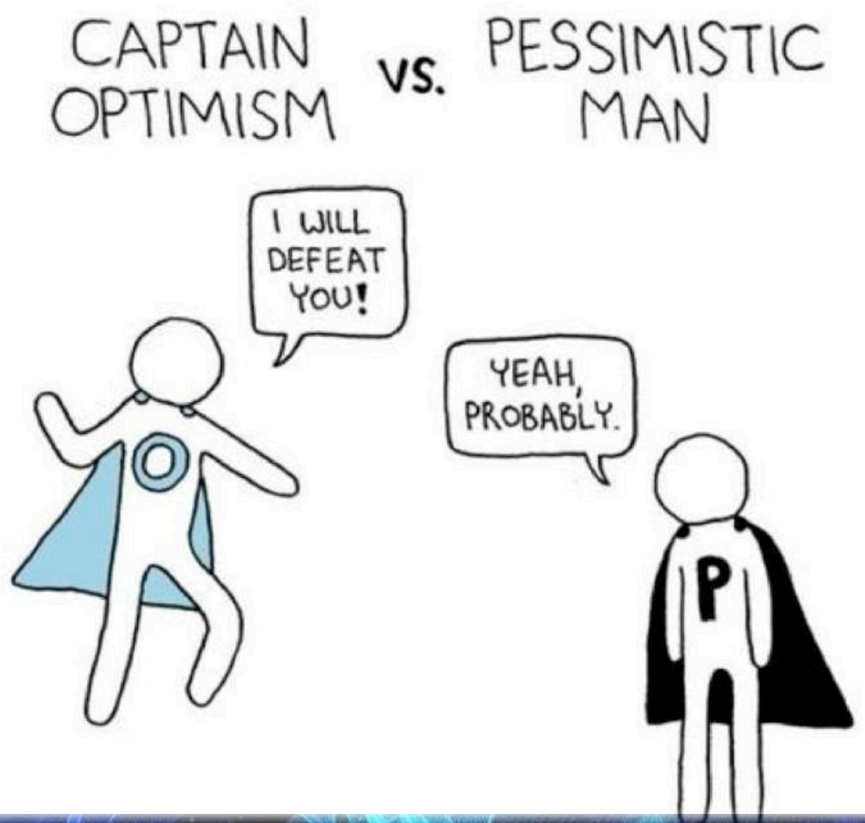
- In the modern control era, control engineers have started dealing explicitely with uncertainty
- The idea of robust control has been playing a fundamental role
- A robust controller garantees performance satisfaction for all possible values of the uncertainty



# Robust control

- In the modern control era, control engineers have started dealing explicitely with uncertainty
- The idea of robust control became fundamental
- A robust controller garantees performance satisfaction for all possible values of the uncertainty

- Or, said differently, the controller is designed to cope with the worst-case scenario
- The resulting design will be inevitably conservative
- This is a pessimistic viewpoint

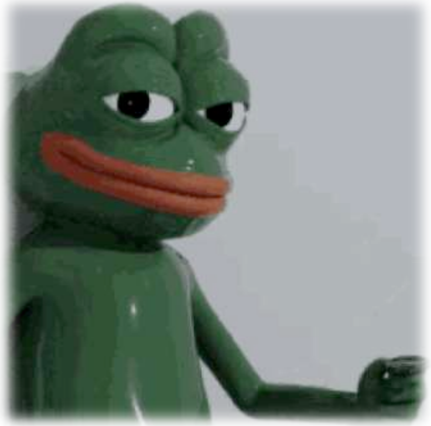


# An optimistic viewpoint to control: probabilistic robustness

*"don't assume the worst-case scenario: it's emotionally draining and probably won't happen anyway"*

Roberto Tempo
CNR IEIIT

- [Tempo, Bai, FD(1997), Calafiore, Campi(2006), Campi,Garatti(2008), Calafiore, FD, Tempo(2011)]



- Probabilistic robustness guarantees that the solution is viable (feasible) in most cases
- In systems & control terms, this translates in accepting some risk that the performance may be violated
- However, the probabilistic formulation is in general even harder
- The main tool: randomized algorithms

# Optimization under uncertainty: the chance constraints set

# Optimization under uncertainty

- Consider an uncertain optimization problem, which amounts at minimizing a linear function under uncertain constraints

- Robust optimization (RO)

$$\min c^\top \theta$$
$$\text{s.t. } f(\theta, w) \leq 0 \quad \forall w \in \mathbb{W}$$


- Chance-constrained optimization (CCO)

$$\min c^\top \theta$$
$$\text{s.t. } Pr_{\mathbb{W}}\{f(\theta, w) \not\leq 0\} \leq \epsilon$$


the parameter $\varepsilon$ is called the violation probability

$$\min \mathbb{E}[f(\theta, w)]$$

# The problem: CCO with linear inequalities

- For simplicity, we will deal here with a set of $n_\ell$ **uncertain linear inequalities**

$$F(w)\theta \leq g(w)$$

- with

$$F(w) = \begin{bmatrix} f_1^\top(w) \\ \vdots \\ f_{n_\ell}^\top(w) \end{bmatrix} \in \mathbb{R}^{n_\ell \times n_\theta}, \quad g(w) = \begin{bmatrix} g_1(w) \\ \vdots \\ g_{n_\ell}(w) \end{bmatrix} \in \mathbb{R}^{n_\ell},$$

- Due to the random nature of the uncertainty each realization of $w$ corresponds to a different set of linear inequalities, giving raise to a corresponding set

$$\mathbb{X}(w) \doteq \{ \, \theta \in \Theta \; : \; F(w)\theta \leq g(w) \, \}$$

# The chance constrained set

- The **probability of violation** of a **given design** $\theta$ is

$$\text{Viol}(\theta) \doteq \Pr_{\mathbb{W}} \{ F(w)\theta \not\leq g(w) \}$$

- The **chance constrained set** of probability $\varepsilon$ is defined as

$$\mathbb{X}_\varepsilon \doteq \{ \theta \in \Theta \ : \ \text{Viol}(\theta) \leq \varepsilon \} \qquad \varepsilon\text{-CCS}$$

- Notice that we consider here joint chance constraints, as opposite to individual chance constraints of the form

$$\theta \in \mathbb{X}^{\ell}_{\varepsilon_\ell} \ \doteq \ \left\{ \theta \in \Theta \ : \ \Pr_{\mathbb{W}} \{ f_\ell(w)^\top \theta \leq g_\ell(w) \} \geq 1 - \varepsilon_\ell \right\}, \quad \ell \in [n_\ell]$$

# The chance constrained set

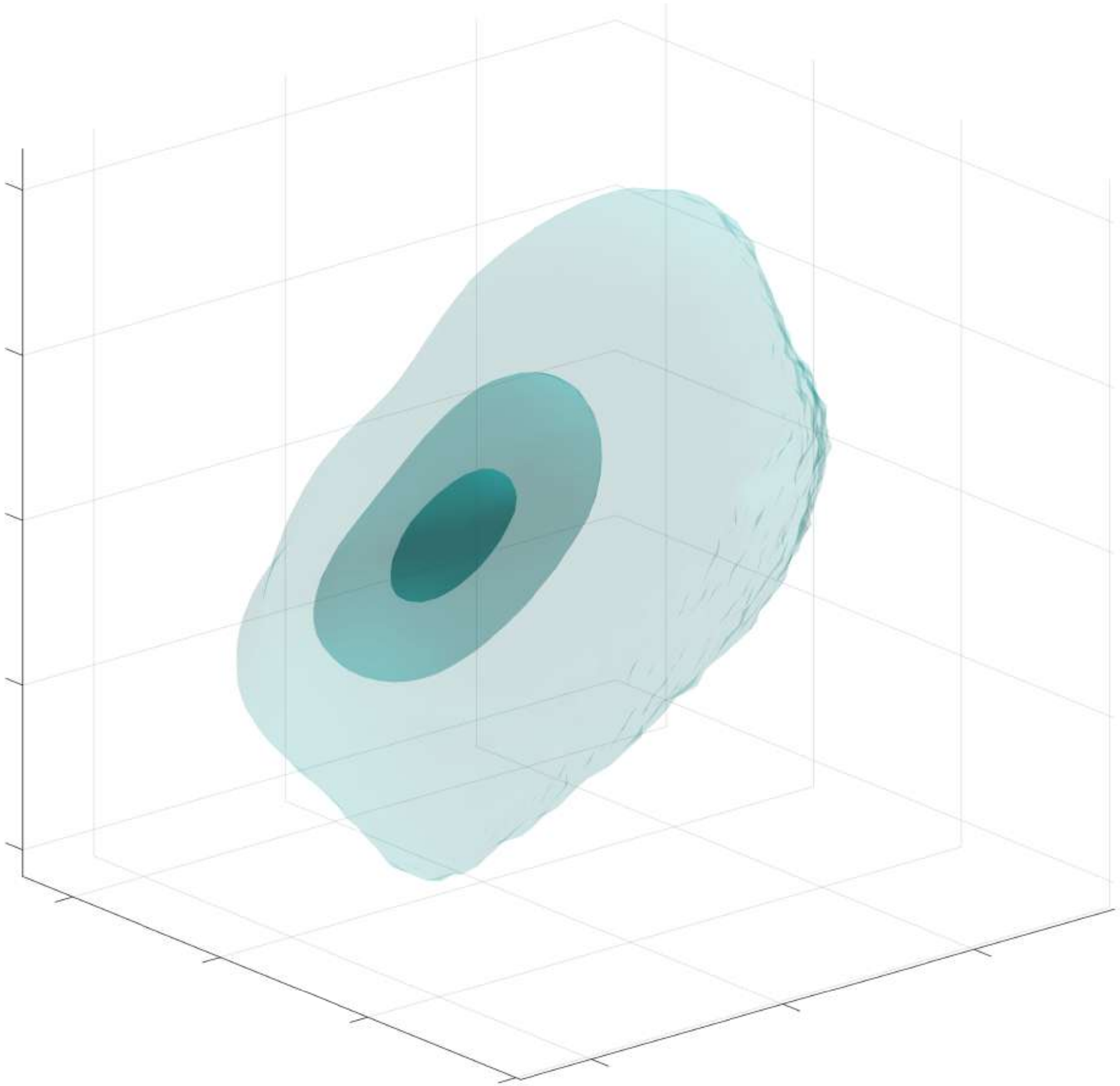
- The $\varepsilon$-CCS is in general nonconvex [Shapiro, Dentcheva, and Ruszczynski(2014)]

**Example 1 (Example of nonconvex $\varepsilon$-CCS)** *To illustrate these inherent difficulties, we consider the following three-dimensional example ($n_\theta = 3$) with $w = \{w_1, w_2\}$, where the first uncertainty $w_1 \in \mathbb{R}^3$ is a three-dimensional normal-distributed random vector with zero mean and covariance matrix*

$$\Sigma = \begin{bmatrix} 4.5 & 2.26 & 1.4 \\ 2.26 & 3.58 & 1.94 \\ 1.4 & 1.94 & 2.19 \end{bmatrix},$$

*and the second uncertainty $w_2 \in \mathbb{R}^3$ is a three-dimensional random vector whose elements are uniformly distributed in the interval $[0, 1]$. The set of viable design parameters is given by $n_\ell = 4$ uncertain linear inequalities of the form*

$$F(w)\theta \leq \mathbf{1}_4, \quad F(w) = \begin{bmatrix} w_1 & w_2 & (2w_1 - w_2) & w_1^2 \end{bmatrix}^\top.$$



Notice eventual convexity behavior [van Ackooij(2015)]

# Our problem

**Problem ( $\varepsilon$-$\mathrm{CCS}$ approximation)** Given the set of linear inequalities

$$F(w)\theta \leq g(w)$$

and a violation parameter $\varepsilon$ , find an inner approximation of the set $\mathbb{X}_\varepsilon$
The approximation should be:
i) simple enough
ii) easily computable

Note that we are interested in approximating the $\varepsilon$ -CCS *per se*,
not in approximating the solution of a CC optimization problem

# Motivations (from systems and control)

Why?

- Why are we interested in directly approximating the CSS?

- **Stochastic Model Predictive Control**: in SMPC we need to solve – online, and so very fast – iterative optimization problems
  - We can reformulate the problem in such a way that, at each step, we solve a problem with different cost functions (depending on your current state) subject to the same CSS
  - If we have "nice" approximations of CSS, we can have efficient algorithms

- **Probabilistic set-membership identification**: in SMI the goal is to identify the set of systems parameters which are compatible with the measurements (the so-called feasible set)
  - In probabilistic SMI, we look for a probabilistic description of the feasibile set under probabilistic assumptions on the noise
  - The probabilistic feasible set is exactly a CSS

- …

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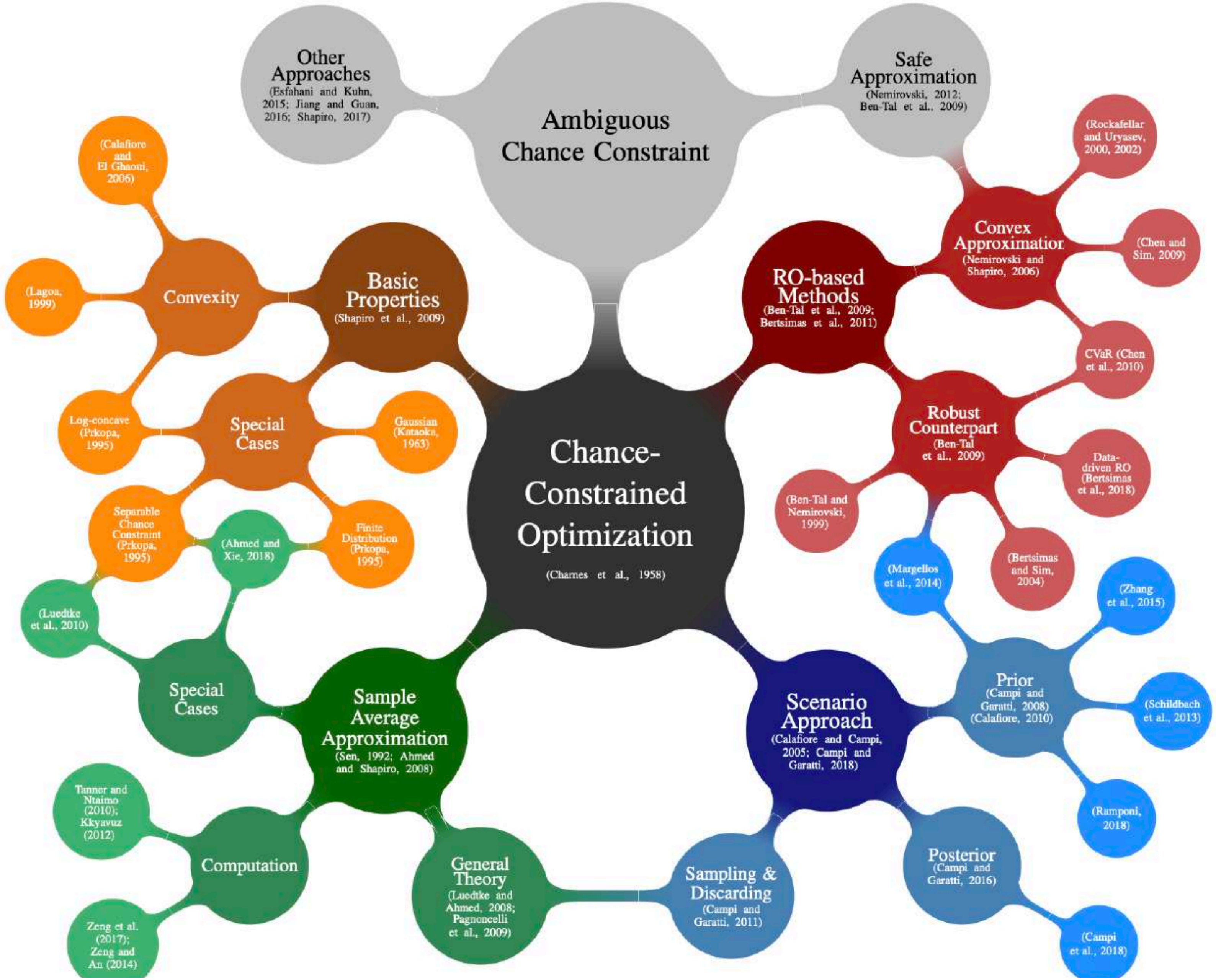
# Approaches to CCO

# Approaches to CCO

- Chance constrained optimization

$$\min_{\theta \in \mathbb{X}_\varepsilon} J(\theta)$$

- Exact techniques: special cases where $\varepsilon$-$\mathrm{CCS}$ is convex and CCO problem admits a unique solution
  - *individual* chance constraints with $w$ Gaussian [Kataoka(1963)]
  - log-concave distribution [Prékopa(1995), Prékopa(1971)]
- Robust techniques: *deterministic* conditions to construct a set $\underline{\mathbb{X}} \subseteq \mathbb{X}_\varepsilon$
  - Chebyshev-like inequalities [Hewing and Zeilinger(2018), Yan et al.(2018)]
  - Robust optimization [Ben-Tal and Nemirovski(1998), Nemirovski and Shapiro(2006)]
  - Conditional Value at Risk (CVaR) [Chen et al.(2010)]
  - Polynomial moments relaxations [Jasour et al.(2015), Lasserre(2017)]
- Sample-based methods
  - discussed next...




Geng Xinbo, Xie Le, "*Data-driven decision making in power systems with probabilistic guarantees: Theory and applications of chance-constrained optimization*", Ann. Reviews in Control

# Sample-based techniques: scenario approach

- We have *N* iid samples of the uncertainty

$$\{w^{(1)}, w^{(2)}, \ldots, w^{(N)}\}$$

- To each sample we associate the following sampled set (scenario)

$$\mathbb{X}(w^{(i)}) = \{ \ \theta \in \Theta \ : \ F(w^{(i)})\theta \le g(w^{(i)}) \ \}$$

- The scenario approach considers the CCO problem and approximates its solution through the following scenario problem

$$\theta^*_{sc} = \arg\min J(\theta)$$
$$\text{subject to } \theta \in \mathbb{X}(w^{(i)}), i \in [N].$$

# Violation probability of scenario

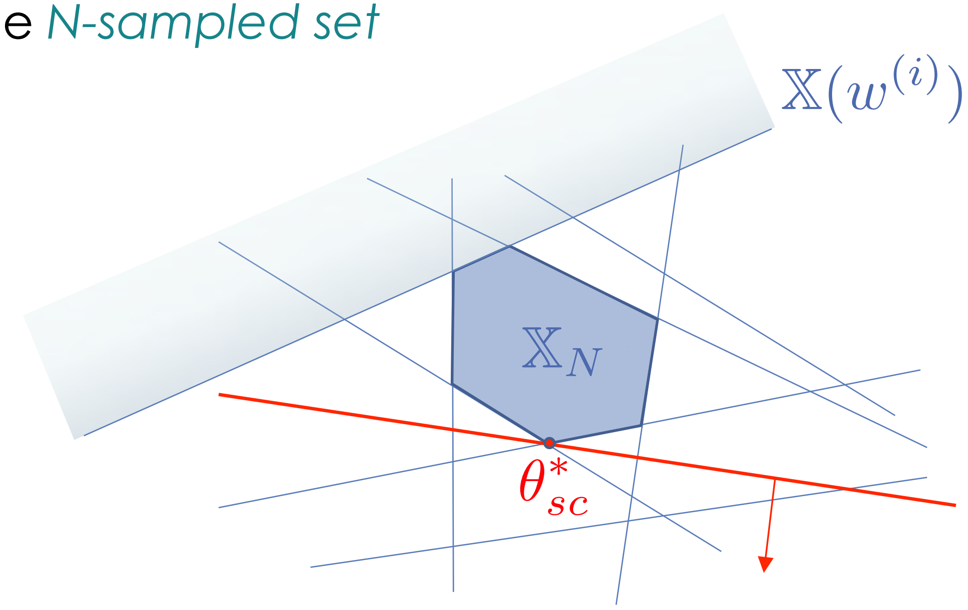
$$\mathbf{B}(k;N,\varepsilon) \doteq \sum_{i=0}^{k} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i}$$

- If J is convex, we have that

$$\text{Pr}_{\mathbb{W}^N} \{\text{Viol}(\theta^*_{sc}) > \varepsilon\} \leq \mathbf{B}(n_\theta - 1; N, \varepsilon)$$

- In a sense, we are approximating the $\varepsilon$-CCS by the *N-sampled set*

$$\mathbb{X}_N \doteq \bigcap_{i=1}^{N} \mathbb{X}(w^{(i)})$$




- But, the probabilistic property above holds only for the optimum $\theta^*_{sc}$ of the scenario program

# Problem: sample-based approximations of CCS

- The results of SO are valid only for the optimal solution, that is we know that

$$\theta^*_{sc} \in \mathbb{X}_\varepsilon$$

but we don't know the N-sampled set is a good approximation of the $\varepsilon$-CCS, i.e. if

$$\mathbb{X}_N \approx \mathbb{X}_\varepsilon ?$$

- Again, this is exactly the problem addressed in this talk:

**Problem ( $\varepsilon$-CCS approximation)** Given the set of linear inequalities

$$F(w)\theta \leq g(w)\}$$

and a violation parameter $\varepsilon$ , find an inner approximation of the set $\mathbb{X}_\varepsilon$
The approximation should be: i) simple enough, ii) easily computable

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# Statistical Learning-Theory based approximations of CCS



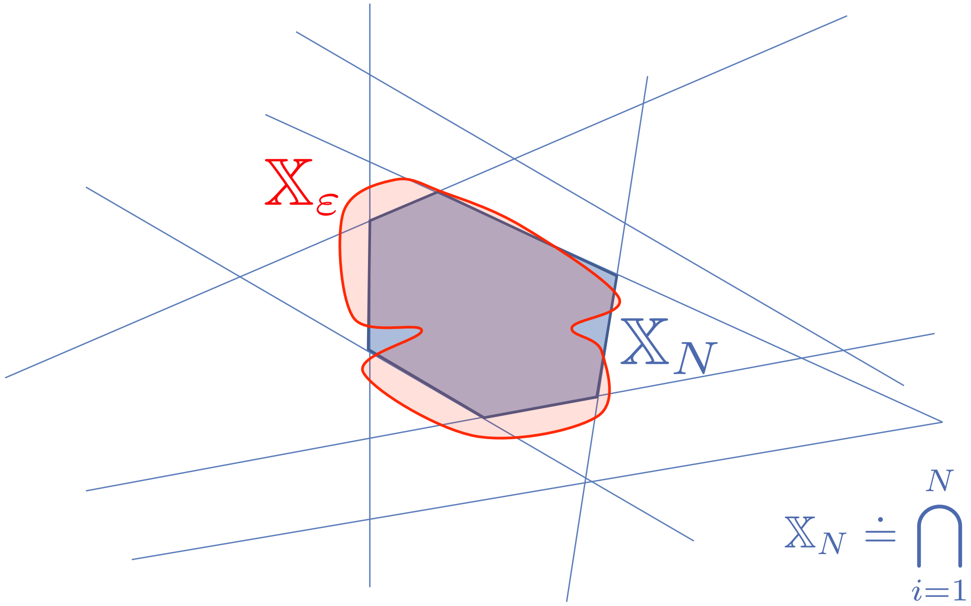

# Learning-theory bound for CCS

- Given probabilistic levels $\delta \in (0,1), \varepsilon \in (0, 0.14)$ if $N \geq N_{LT}$, with

$$N_{LT} \doteq \frac{4.1}{\varepsilon}\left(\ln \frac{21.64}{\delta} + 4.39 n_\theta \log_2\left(\frac{8en_\ell}{\varepsilon}\right)\right)$$

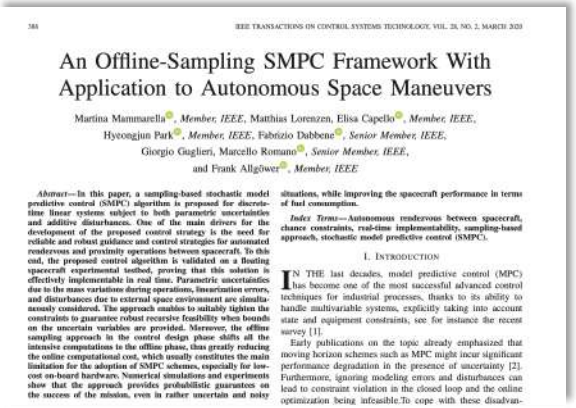
- then

$$\mathrm{Pr}_{\mathbb{W}^N}\left\{\mathbb{X}_N \subseteq \mathbb{X}_\varepsilon\right\} \geq 1 - \delta$$




$$\mathbb{X}_N \doteq \bigcap_{i=1}^{N} \mathbb{X}(w^{(i)})$$

# SMPC for automated rendezvous


IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY, VOL. 28, NO. 2, MARCH 2020

An Offline-Sampling SMPC Framework With Application to Autonomous Space Maneuvers

Martina Mammarella, Member, IEEE, Matthias Lorenzen, Elisa Capello, Member, IEEE, Hyeongjun Park, Member, IEEE, Fabrizio Dabbene, Senior Member, IEEE, Giorgio Guglieri, Marcello Romano, Senior Member, IEEE, and Frank Allgöwer, Member, IEEE

*Abstract*—In this paper, a sampling-based stochastic model predictive control (SMPC) algorithm is proposed for discrete-time linear systems subject to both parametric uncertainties and additive disturbances. One of the main drivers for the development of the proposed control strategy is the need for reliable and robust guidance and control strategies for automated rendezvous and proximity operations between spacecraft. To this end, the proposed control algorithm is validated on a floating spacecraft experimental testbed, proving that this solution is effectively implementable in real time. Parametric uncertainties due to the mass variations during operations, linearization errors, and disturbances due to external space environment are simultaneously considered. The approach enables to suitably tighten the constraints to guarantee robust recursive feasibility when bounds on the uncertain variables are provided. Moreover, the offline sampling approach in the control design phase shifts all the intensive computations to the offline phase, thus greatly reducing the online computational cost, which usually constitutes the main limitation for the adoption of SMPC schemes, especially for low-cost on-board hardware. Numerical simulations and experiments show that the approach provides probabilistic guarantees on the success of the mission, even in rather uncertain and noisy situations, while improving the spacecraft performance in terms of fuel consumption.

*Index Terms*—Autonomous rendezvous between spacecraft, chance constraints, real-time implementability, sampling-based approach, stochastic model predictive control (SMPC).

I. Introduction

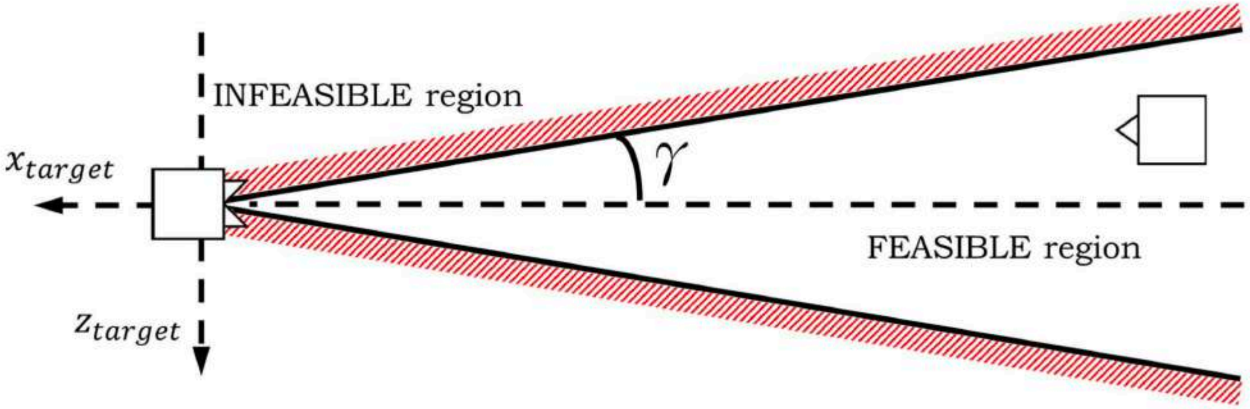
In THE last decades, model predictive control (MPC) has become one of the most successful advanced control techniques for industrial processes, thanks to its ability to handle multivariable systems, explicitly taking into account state and equipment constraints; see for instance the recent survey [1].

Early publications on the topic already emphasized that moving horizon schemes such as MPC might incur significant performance degradation in the presence of uncertainty [2]. Furthermore, ignoring modeling errors and disturbances can lead to constraint violation in the closed loop and the online optimization being infeasible. To cope with these disadvan-

- We consider systems of the form

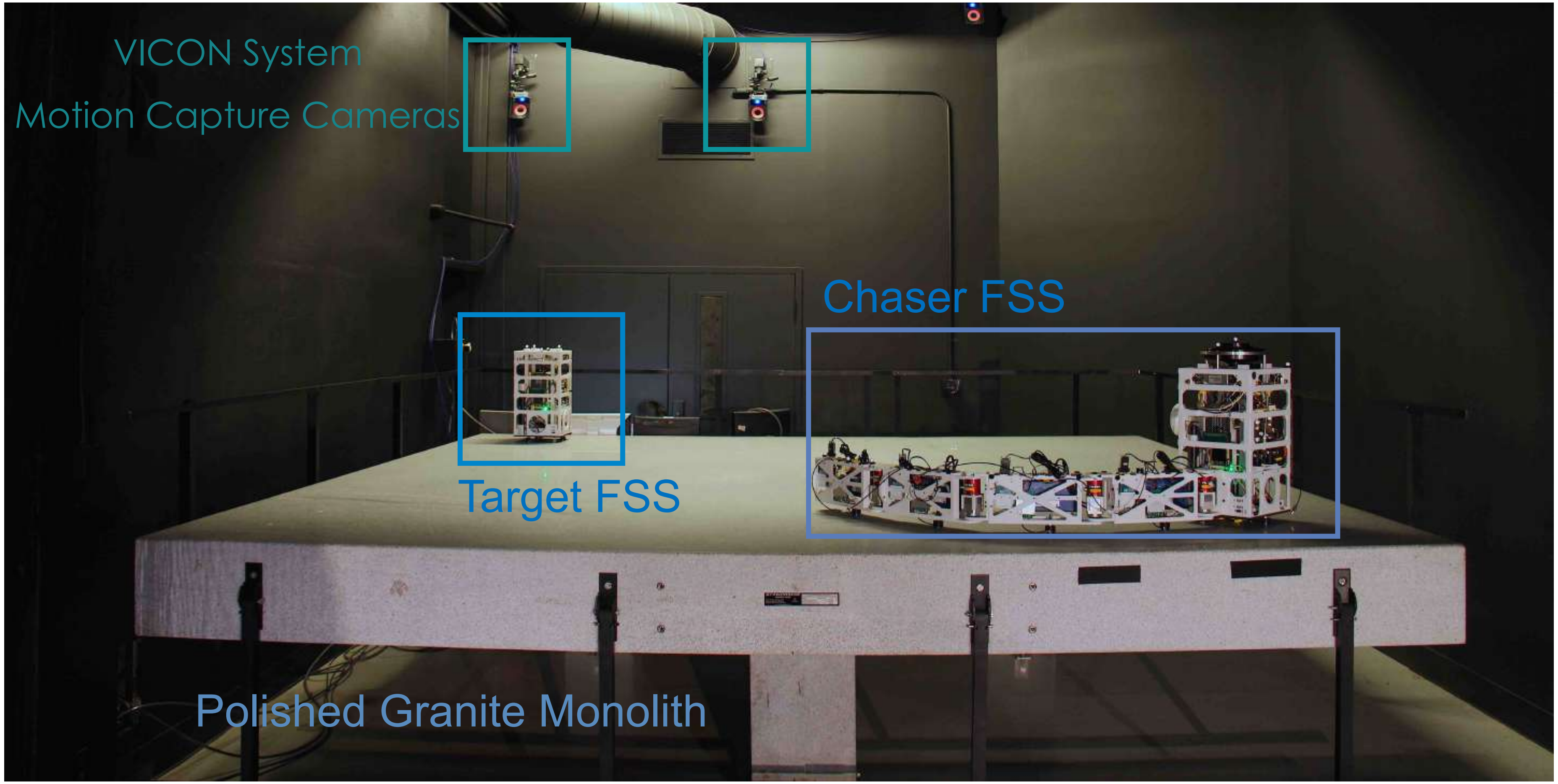
$$x_{k+1} = A(q_k)x_k + B(q_k)u_k + B_w(q_k)w_k$$

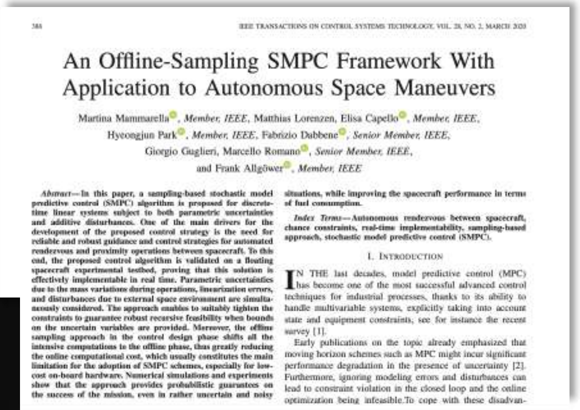
- Our sample-based approach guarantees robust feasibility, aymptotic convergence in probability, efficient online implementation
- The approach is successfully applied to the development of flyable SMPC schemes for automated rendezvous and proximity operations between spacecraft



# SMPC for automated rendezvous

- The developed techniques were tested on a testbed at NPS



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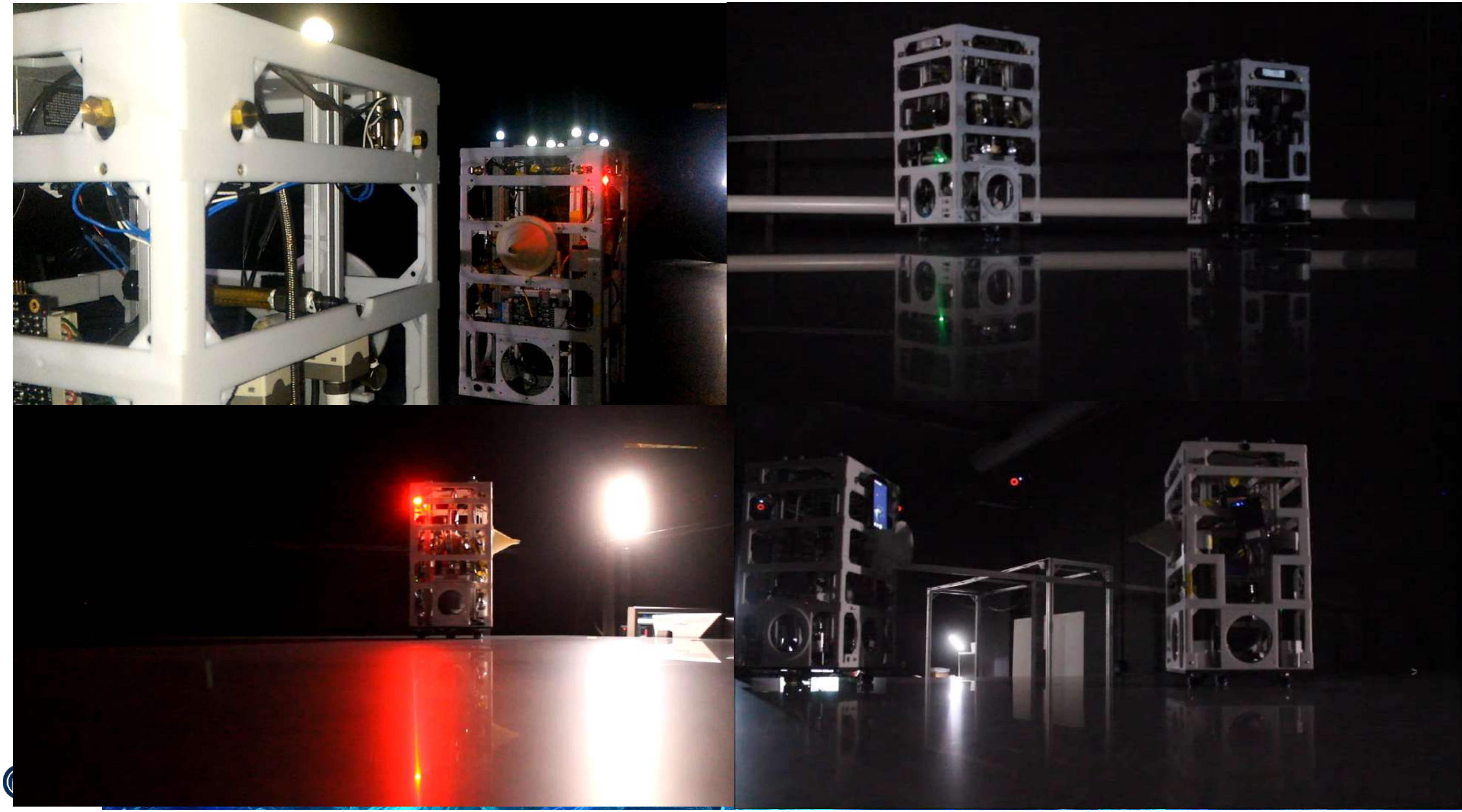
Naval Postgraduate School, Monterey, CA
POSEIDYN Air Bearing Testbed



# LT approximations – pro and cons

- The LT approaches based on sampled approximations have proved very effective
- In SMPC, the possibility of performing the “heavy” computations offline allowed to derive computationally efficient implementations
- Also, one can perform ofline “constraint pruning” to lower the number of constraints
- However, in general, the number of constraints which we have to deal with may still be prohibitive
  - even for a moderately sized MPC problem with 5 states, 2 inputs, prediction horizon T = 10, simple interval constraints on states and inputs, and for probabilistic parameters $\varepsilon = 0.05$, $\delta = 10{-}6$, we get more than 1.6 million linear inequalities (before pruning)

- We would like to have a method which is ”tunable”, depending on our computational power
- This method we propose is based on probabilistic scaling

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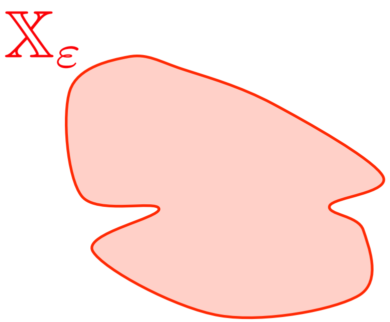
# Probabilistic scaling

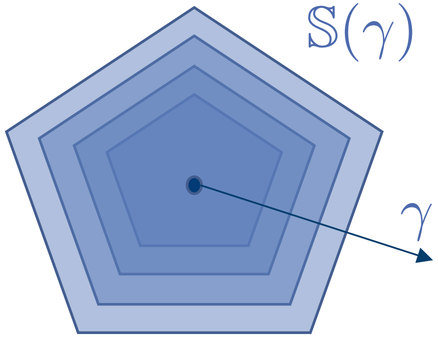



# Probabilistic scaling



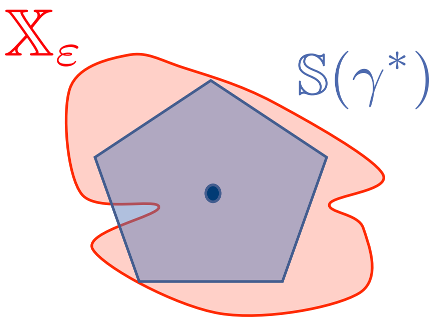

- **AIM**: to approximate the $\varepsilon$-CCS

$$\mathbb{X}_\varepsilon \doteq \{\, \theta \in \Theta \;:\; \text{Viol}(\theta) \leq \varepsilon \,\}$$




- **IDEA**: start with simple approximating sets (Scalable SAS)

$$\mathbb{S}(\gamma) = \theta_c \oplus \gamma\mathbb{S}$$




- We scale the set so that it constitutes a good approximation of the CCS
- That is, we look for an optimal scaling factor $\gamma^*$ so that

$$\text{Pr}\{\mathbb{S}(\gamma^*) \subseteq \mathbb{X}_\varepsilon\} \geq 1 - \beta$$

# Probabilistic scaling

- Assume a Scalable SAS $\mathbb{S}(\gamma)$ is available
- We propose a sample-based procedure: we assume that $N_\gamma$ iid samples from $\Pr_{\mathbb{W}}$ are available

$$\{w^{(1)}, \ldots, w^{(N_\gamma)}\}$$

- Based on these, we show how to obtain a scalar $\bar{\gamma}$ such that

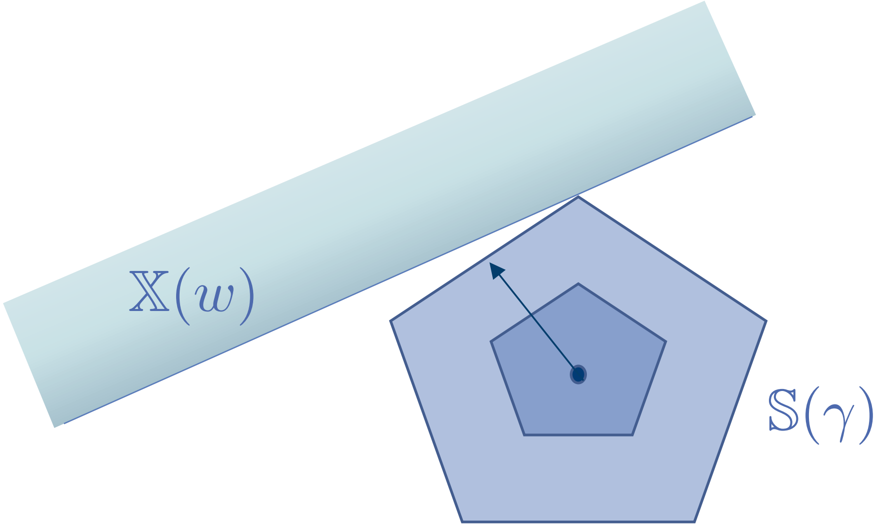
$$\Pr_{\mathbb{W}^{N_\gamma}} \{\mathbb{S}(\bar{\gamma}) \subseteq \mathbb{X}_\varepsilon\} \geq 1 - \delta$$

# Scaling factor

**Definition (Scaling factor):** Given a Scalable SAS $\mathbb{S}(\gamma)$, with center $\theta_c$ and shape $\mathbb{S}$, we define the scaling factor of $\mathbb{S}(\gamma)$ relative to the realization $w \in \mathbb{W}$ as

$$\gamma(w) \doteq \begin{cases} 0 & \text{if } \theta_c \notin \mathbb{X}(w) \\ \max\limits_{\mathbb{S}(\gamma)\subseteq\mathbb{X}(w)} \gamma & \text{otherwise.} \end{cases}$$

That is, $\gamma(w)$ is the maximal scaling that can be applied to the SAS around its center so that $\mathbb{S}(\gamma) \subseteq \mathbb{X}(w)$



# Probabilistic scaling computation

**Algorithm 1** Probabilistic SAS Scaling

1: Given a candidate Scalable SAS $\mathbb{S}(\gamma)$, and probability levels $\varepsilon$ and $\delta$, choose

$$N_\gamma \geq \frac{7.47}{\varepsilon} \ln \frac{1}{\delta} \quad \text{and} \quad r = \left\lfloor \frac{\varepsilon N_\gamma}{2} \right\rfloor. \qquad (15)$$

2: Draw $N_\gamma$ samples of the uncertainty $w^{(1)}, \ldots, w^{(N_\gamma)}$.

3: **for** $i = 1$ to $N_\gamma$ **do**

4: Solve the optimization problem

$$\gamma_i \doteq \max_{\mathbb{S}(\gamma) \subseteq \mathbb{X}(w^{(i)})} \gamma. \qquad (16)$$

5: **end for**

6: Return $\bar{\gamma} = \gamma_{1+r:N_\gamma}$, the $(1+r)$-th smallest value of $\gamma_i$.

**Theorem:**

$$\Pr_{\mathbb{W}^{N_\gamma}} \{\mathbb{S}(\bar{\gamma}) \subseteq \mathbb{X}_\varepsilon\} \geq 1 - \delta$$

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# Candidate SAS

# Candidate SAS: Sampled-polytope

- The first natural candidate is clearly an *N*-sampled set
  - Draw $N_S$ design samples $\{\tilde{w}^{(1)}, \dots, \tilde{w}^{(N_S)}\}$ and build

$$\mathbb{X}_{N_S} = \bigcap_{j=1}^{N_S} \mathbb{X}(\tilde{w}^{(j)})$$

- However, in this case, the number $N_S$ is not given by the LT bound, but it is a design parameter
  - We could choose $N_S$ as the number of constraints compatible with our computational power
  - E.g., in SMPC, it could be the number of constraints we can process online in one step

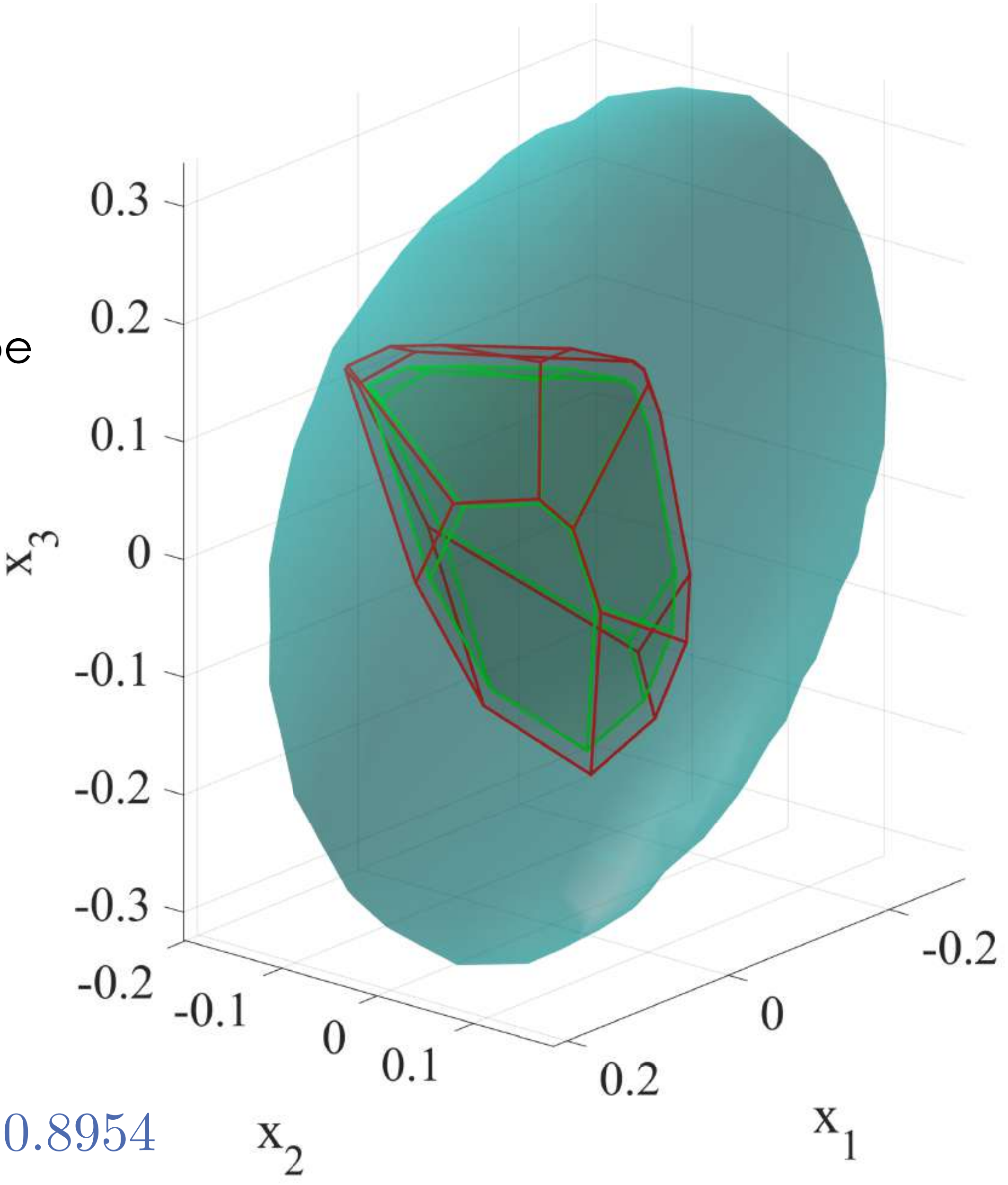
- The center of the set may be chosen as the Chebichev center (or the analytic center in case of linear inequalities)

# Example

Probabilistic scaling approximation of the $\varepsilon$-CCS

Scaling procedure applied to a sampled-polytope with $N_S = 100$

Initial set is depicted in red, the scaled in green



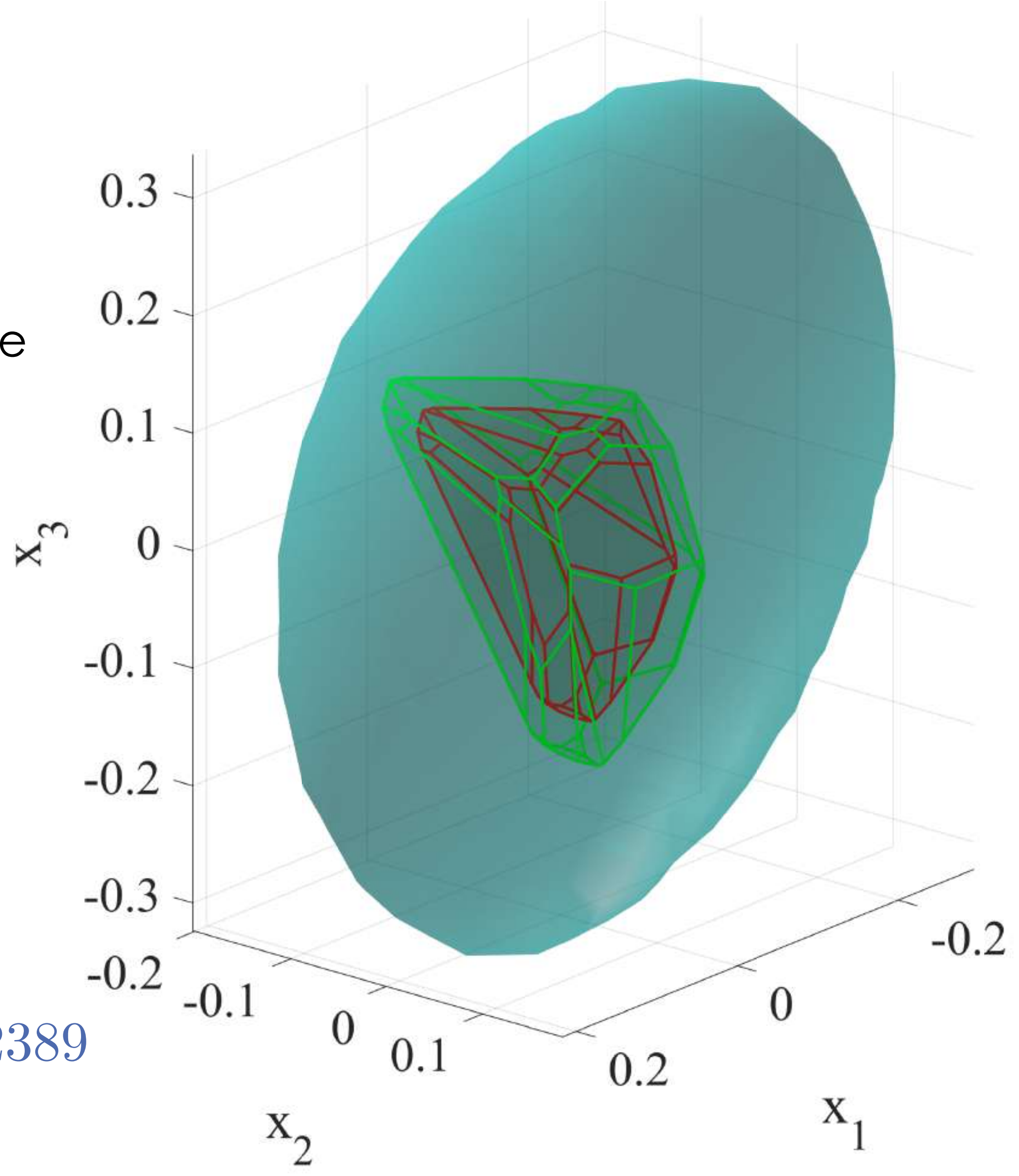

$\bar{\gamma} = 0.8954$

# Example

Probabilistic scaling approximation of the $\varepsilon$-CCS

Scaling procedure applied to a sampled-polytope with $N_S = 1,000$

Initial set is depicted in red, the scaled in green



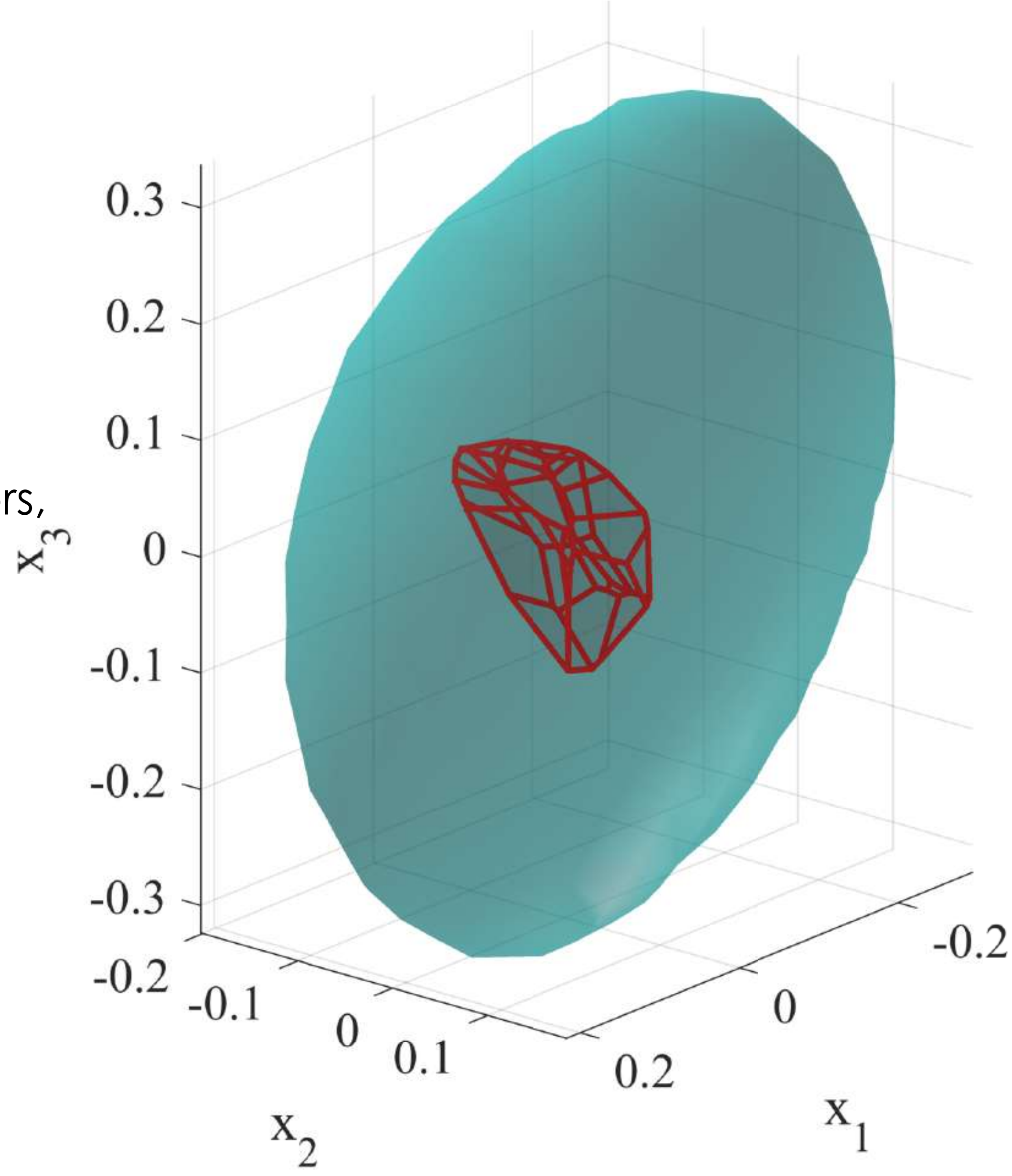

$\bar{\gamma} = 1.2389$

# Example

Probabilistic scaling approximation of the $\varepsilon$-CCS

Approximation obtained by direct application of the LT bound (52,044 linear inequalities)

Note that, in this case, to avoid out-of-memory errors, a pruning procedure was necessary

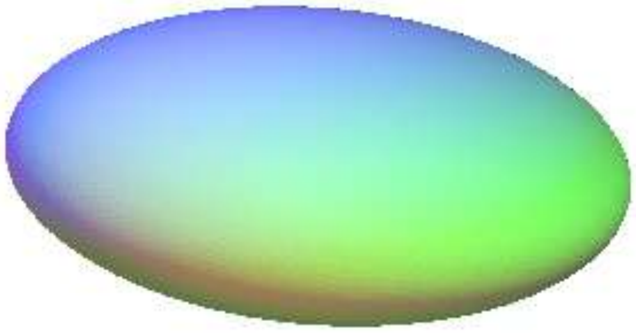


# Candidate SAS: $\ell_p$-norm based sets

- We define norm-based SAS of the form

$$\mathbb{S}_{\ell_p}(\gamma) \doteq \theta_c \oplus \gamma H \mathbb{B}_p^s$$

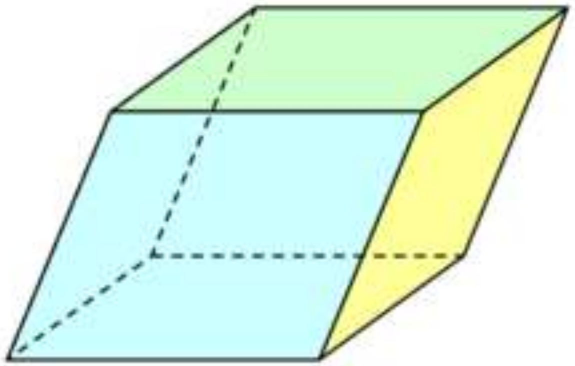
where $\mathbb{B}_p^s$ is the unit ball in the $\ell_p$-norm in $\mathbb{R}^s$ with $s \geq n_\theta$ and $H \in \mathbb{R}^{n_\theta \times s}$ is a shape matrix



ellipsoid

$p = 2, s = n_\theta = 3,$

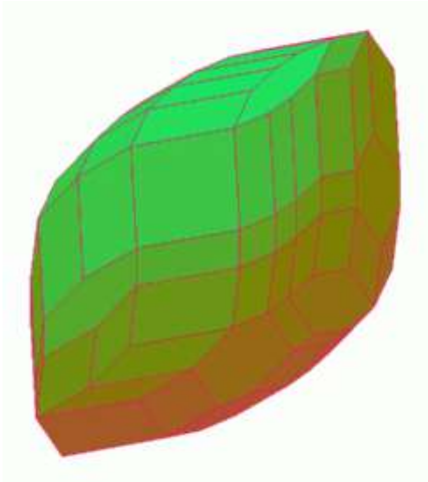
$H = H^T \succ 0$



parallelotope

$p = 1, s = n_\theta = 3,$

$H = H^T \succ 0$



zonotope

$p = \infty, s = 10, n_\theta = 3,$

$H \in \mathbb{R}^{10,3}$

# $\ell_p$-norm based sets – scaling factor computation

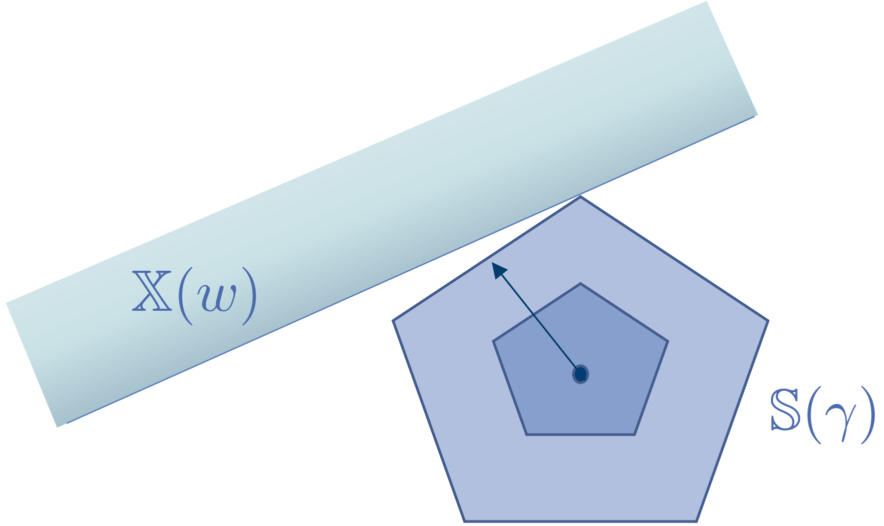
**Theorem 2 (Scaling factor for norm-based SAS)** *Given a norm-based SAS $\mathbb{S}(\gamma)$ as in (20), and a realization $w \in \mathbb{W}$, the scaling factor $\gamma(w)$ can be computed as*

$$\gamma(w) = \min_{\ell \in [n_\ell]} \gamma_\ell(w),$$

*with $\gamma_\ell(w)$, $\ell \in [n_\ell]$, given by*

$$\gamma_\ell(w) = \begin{cases} 0 & \text{if } \tau_\ell(w) < 0, \\ \infty & \text{if } \tau_\ell(w) \geq 0 \text{ and } \rho_\ell(w) = 0, \\ \dfrac{\tau_\ell(w)}{\rho_\ell(w)} & \text{if } \tau_\ell(w) \geq 0 \text{ and } \rho_\ell(w) > 0, \end{cases}$$

*where $\tau_\ell(w) \doteq g_\ell(w) - f_\ell^T(w)\theta_c$ and $\rho_\ell(w) \doteq \|H^T f_\ell(w)\|_{p^*}$, with $\|\cdot\|_{p^*}$ being the dual norm of $\|\cdot\|_p$.*

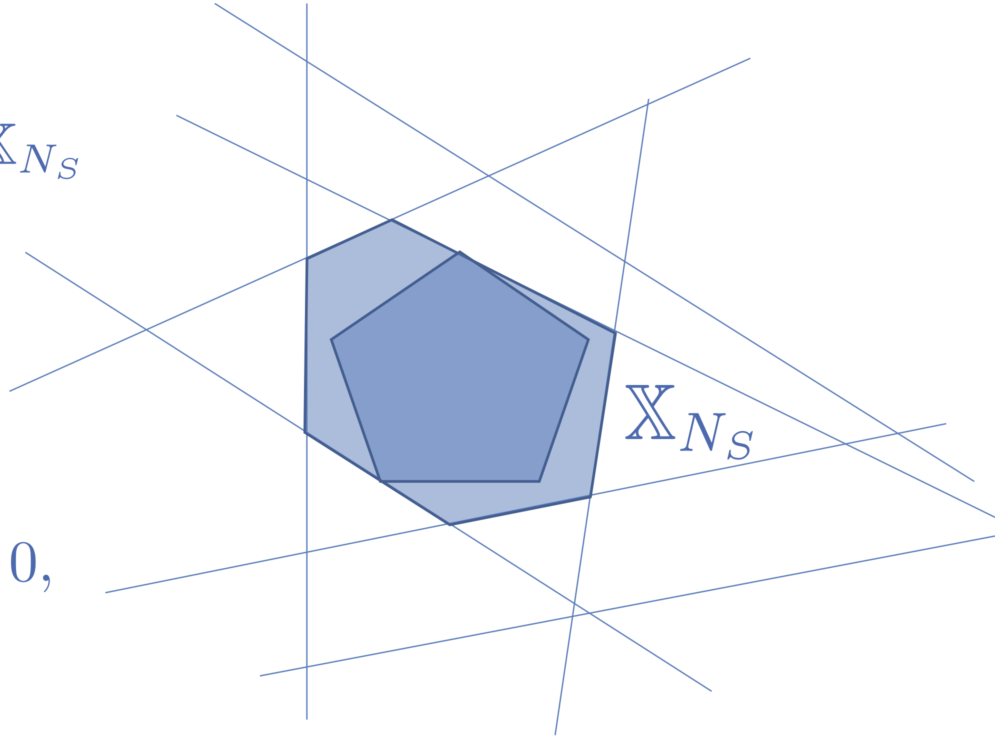


# Construction of a candidate $\ell_p$-norm based set

- We can start again from a (design) *N-sampled set* $\mathbb{X}_{N_S}$
- We compute the largest norm-based set contained in $\mathbb{X}_{N_S}$

$$\max_{\theta_c, H} \ \mathrm{Vol}_p(H)$$

$$\text{subject to } \theta_c \oplus H\mathbb{B}_p^s \subseteq \mathbb{X}_{N_S}$$

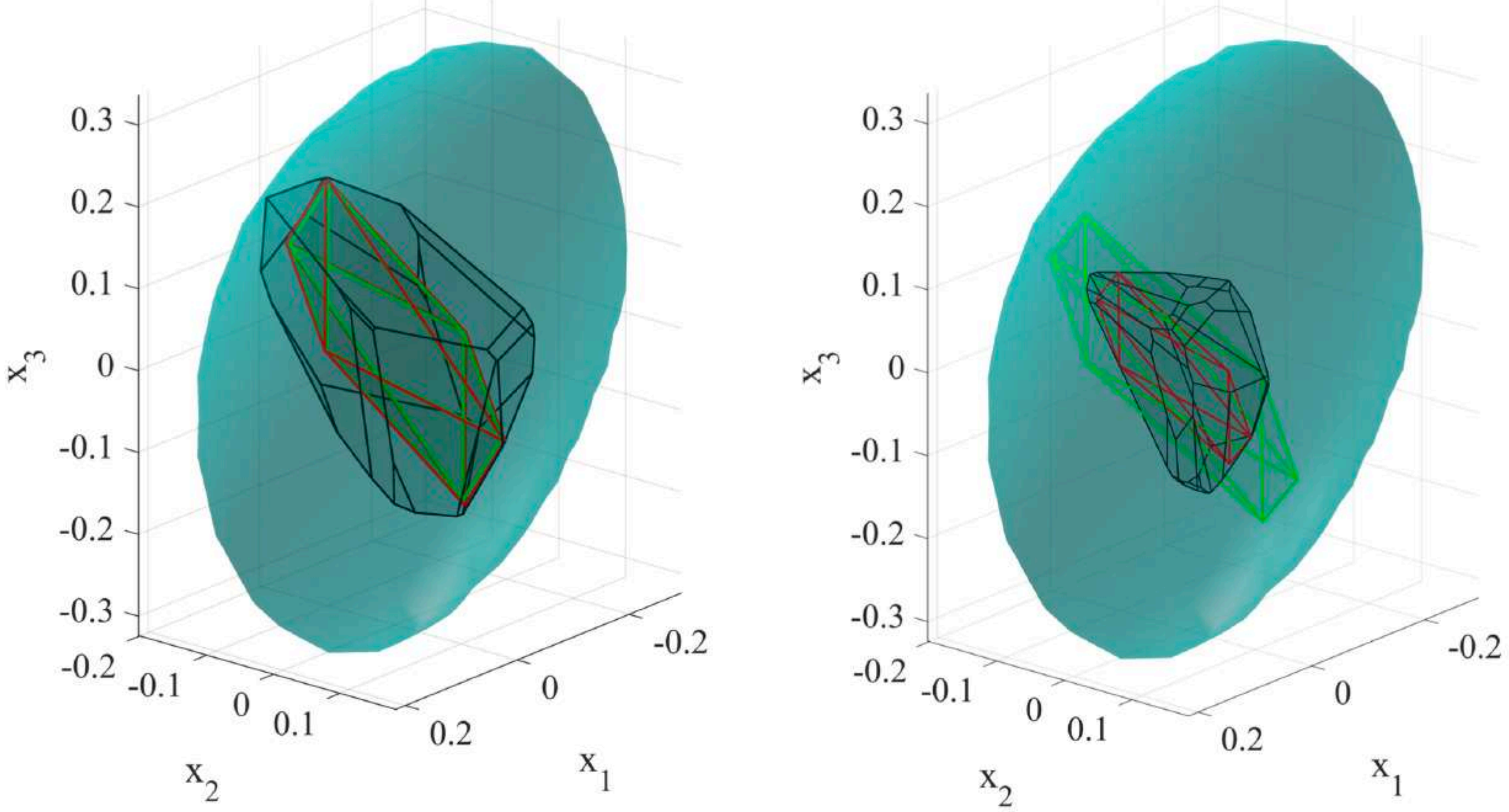



- This problem is equivalent to

$$\begin{aligned} \min_{\theta_c, H} \quad & -\mathrm{Vol}_p(H) \\ \text{s.t.} \quad & f_\ell^T(\tilde{w}^{(j)})\theta_c + \|H^T f_\ell(\tilde{w}^{(j)})\|_{p^*} - g_\ell(\tilde{w}^{(j)}) \leq 0, \\ & \ell \in [n_\ell], \ j \in [N_S] \end{aligned}$$

# $\mathbb{S}_{\ell_1}$ SAS

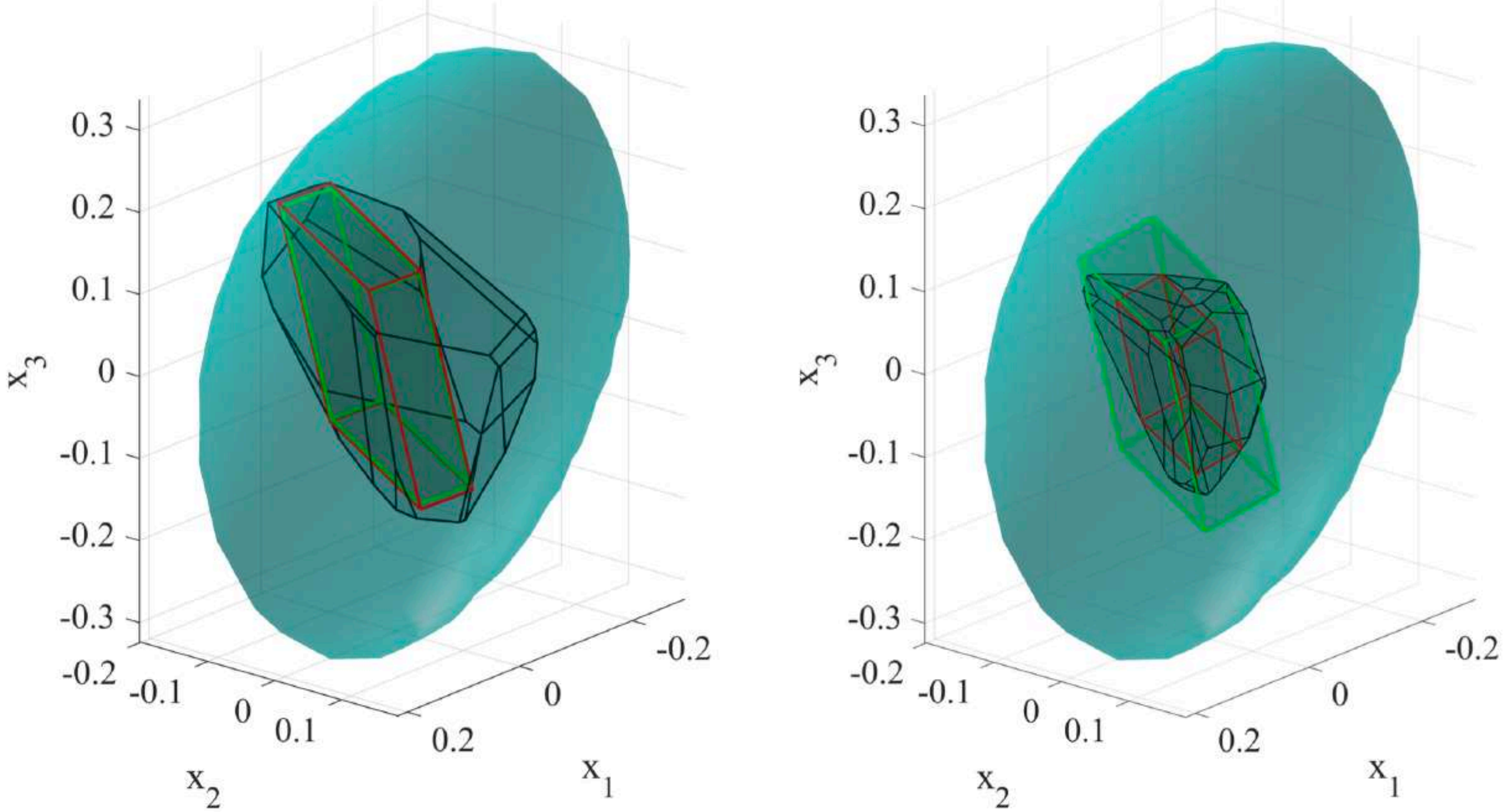
$N_S = 100$ $\quad$ $N_S = 1,000$




(a) $\gamma = 0.9701$ $\quad$ (b) $\gamma = 1.5995$

# $\mathbb{S}_{\ell_\infty}$ SAS

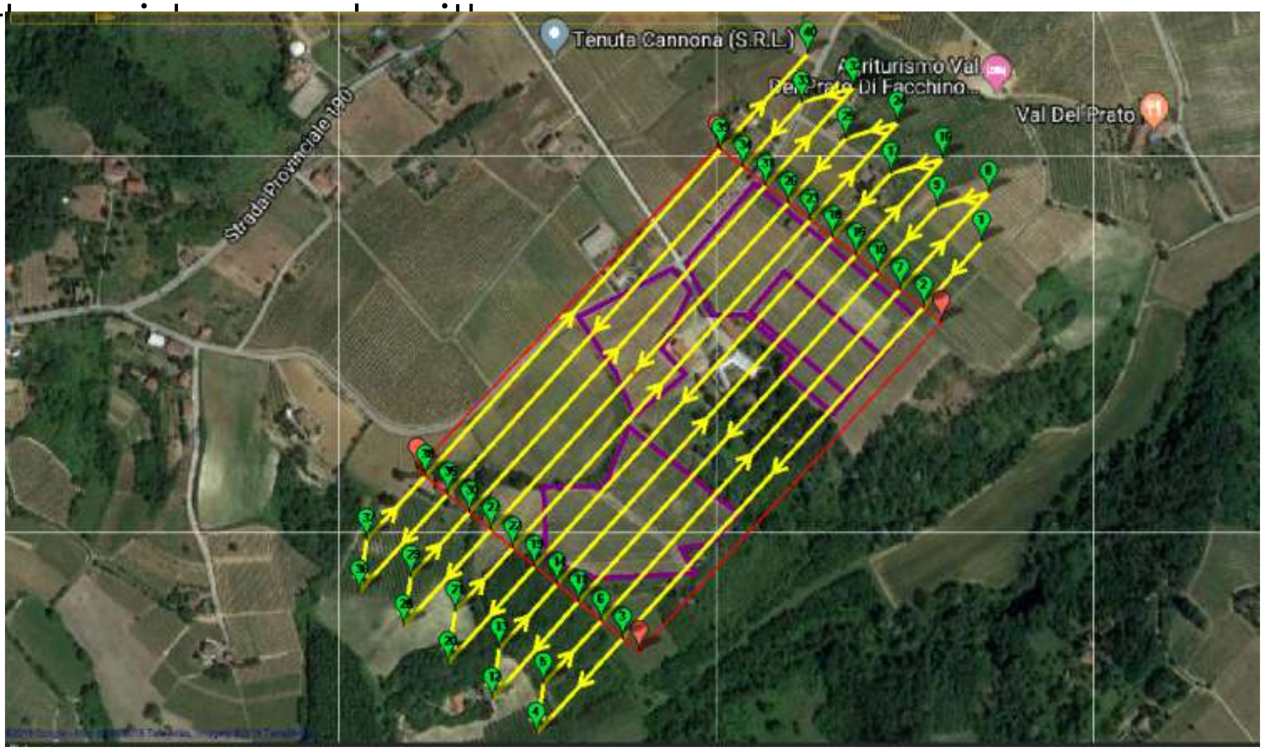
$N_S = 100$ $\qquad$ $N_S = 1,000$




(c) $\gamma = 0.9696$ $\qquad$ (d) $\gamma = 1.5736$

# Application: UAV SMPC tracking control

- The selected application involves a fixed-wing UAV performing a monitoring mission over a sloped Dolcetto vineyard at Carpeneto, Alessandria, Italy ($44^\circ 40' 55.6''$ N, $8^\circ 37' 28.1''$ E)
- The main objective is to provide proper control capabilities to the UAV to guarantee a fixed relative altitude with respect to the terrain of 150 m while following the desired optimal path defined by t

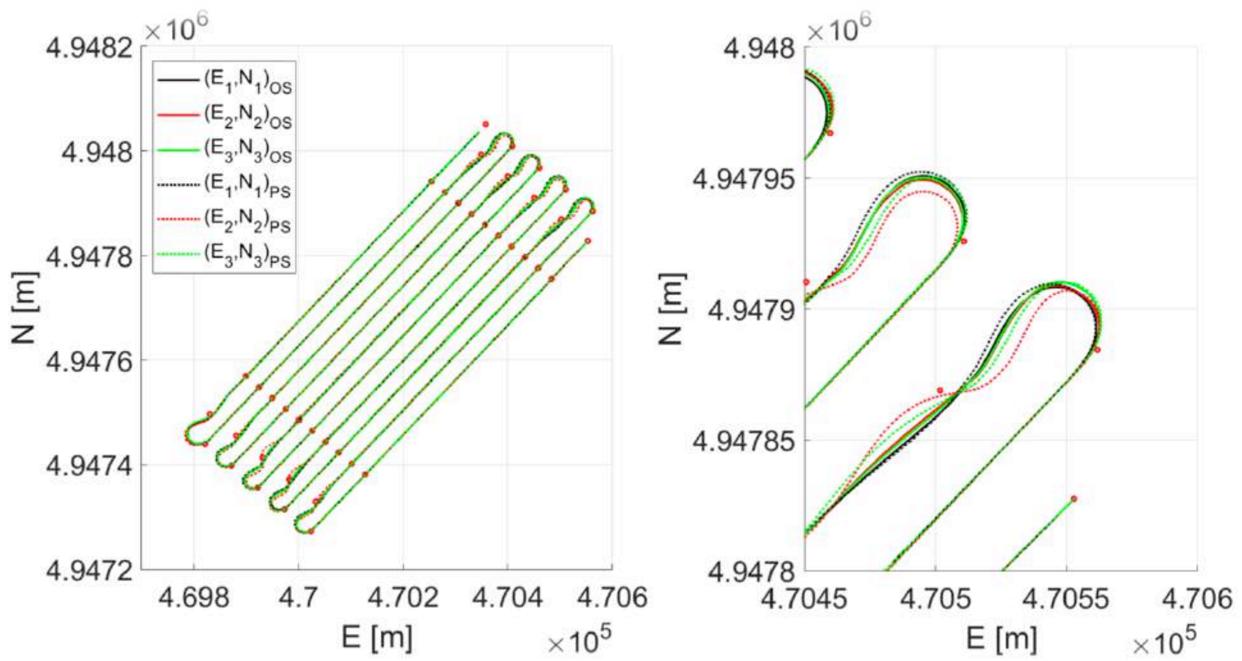


# Simulation: UAV SMPC tracking control

- Prediction horizon T=15
- Offline-sampling approach: 20,604 constraints
- 1-norm based approach: 107 constraints

MAXIMUM AND AVERAGE ONLINE COMPUTATIONAL COST.

| n. | $t_{cMAX_{OS}}$ | $t_{cAVG_{OS}}$ | $t_{cMAX_{PS}}$ | $t_{cAVG_{PS}}$ |
|---|---|---|---|---|
| 1 | 2.0959 | 0.4178 | 0.0966 | 0.0087 |
| 2 | 2.9411 | 0.5626 | 0.7221 | 0.0190 |
| 3 | 2.1497 | 0.5434 | 0.2628 | 0.0086 |




Thank you

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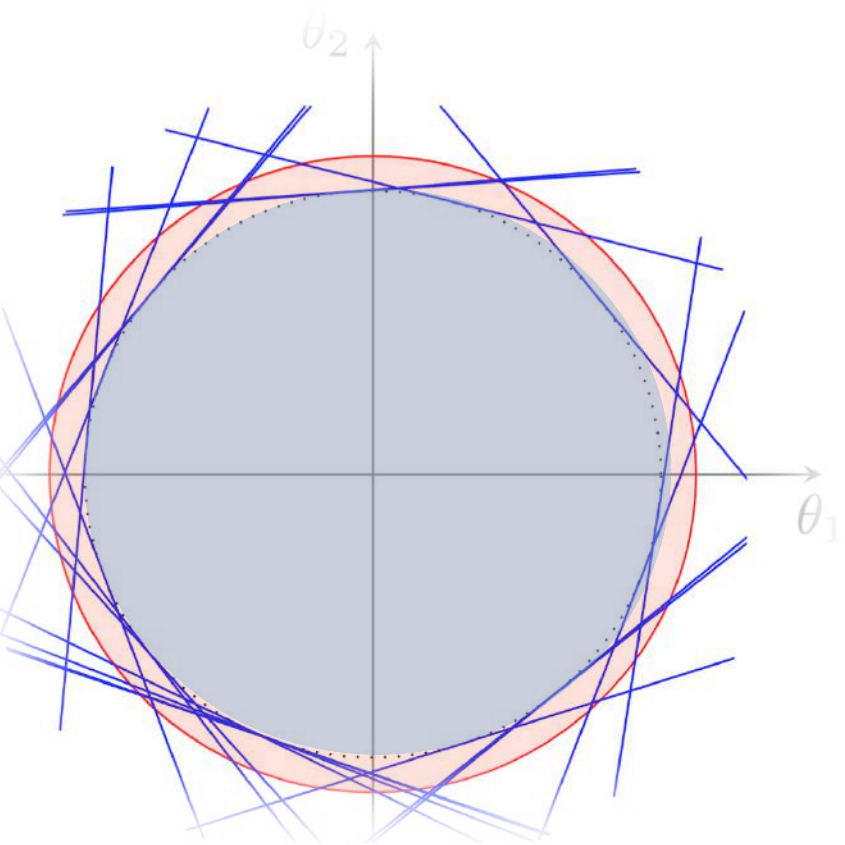
# PAC-based scaling

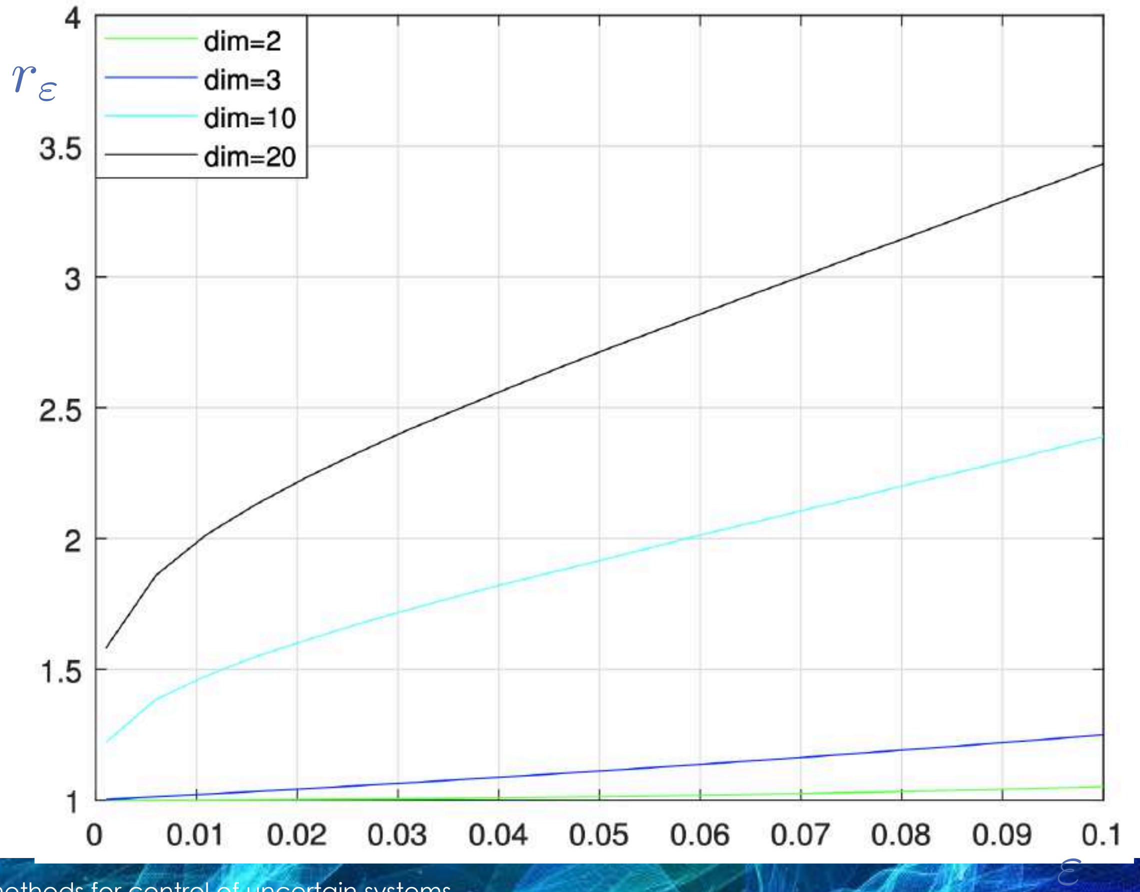
# A simple but interesting example

- Suppose that we generate *N* random linear constraints tangent to points drawn from a uniform probability distribution on the surface of the unit hypersphere
- Suppose that the unit hypersphere constitutes an initial approximation
- It is possible to scale this initial geometry around its center (the origin) to obtain an inner approximation $\mathbb{S}(\gamma) \doteq x_c \oplus \gamma\mathbb{S}$ of $\mathbb{X}_\varepsilon$ with a given level of confidence
- However, this scaling scheme will always provide as a result the unit hypersphere
- On the other hand, the true $\mathbb{X}_\varepsilon$ for $\varepsilon = 0.15$ is larger



# Enlargement factor

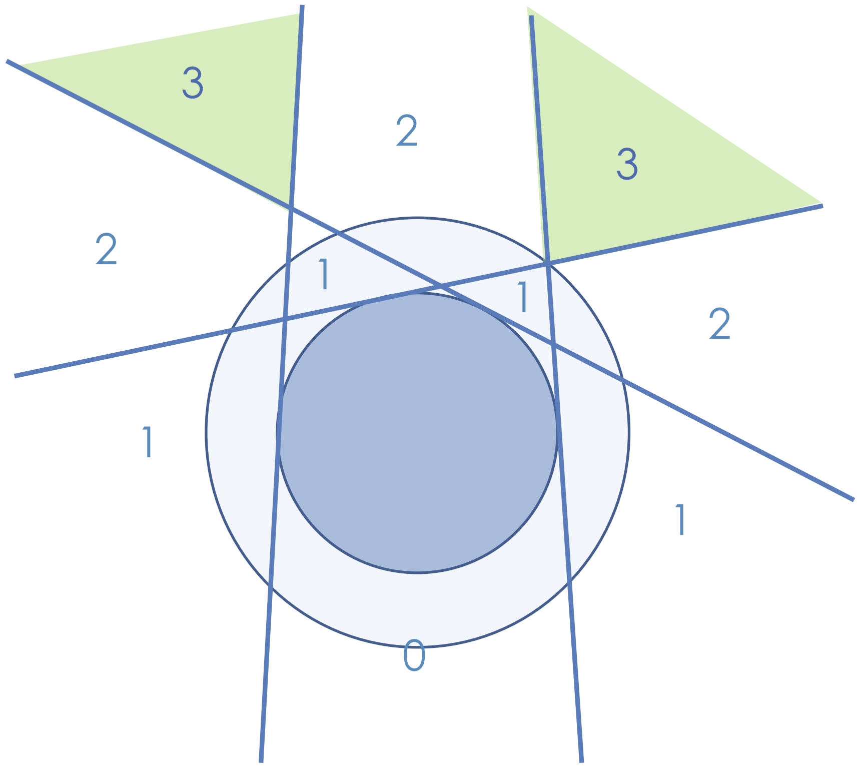
- For this simple case, the radius corresponding to the CCS can be computed in an exact way using some transcendental functions
- The resulting radius turns out to be significantly larger than one



# Pack-based strategies

- We show that a better scaling may be computed using pack-based strategies

- The idea is illustrated in the figure
- The linear constraints divide the plane into regions with different number of violated constraints
- We notice that larger scale factors can be obtained if one scales the unit-circle until it touches the green regions.
- More generally, if only regions in which more than a given number of constraints are violated are considered, then larger scale factors could be contemplated.
- However, this approach has to be designed in such a way that probabilistic guarantees are given



# Pack-based strategies

- Pack-based strategies proved very powerful, but with a significative increase in computational cost
- This is not surprising, since we are aiming at approximating an NP-hard problem
- We have a “nice” tradeoff between conservativeness/computational cost