Exact discretization methods for Multistage Stochastic Linear Problem

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Robustness and Resilience in SO and SL workshop

Ettore Majorana Foundation, Erice

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Multistage stochastic linear programming (MSLP)

$$
\begin{aligned}
\min_{(\boldsymbol{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \boldsymbol{c}_t^\top \boldsymbol{x}_t\Big] & \\
\text{s.t.} \quad & \boldsymbol{A}_t\boldsymbol{x}_t + \boldsymbol{B}_t\boldsymbol{x}_{t-1} \leqslant \boldsymbol{b}_t & \forall t \in [T] \\
& \sigma(\boldsymbol{x}_t) \subset \sigma(\boldsymbol{c}_\tau, \boldsymbol{A}_\tau, \boldsymbol{B}_\tau, \boldsymbol{b}_\tau)_{\tau\leqslant t} & \forall t \in [T] \\
& \boldsymbol{x}_0 \equiv x_0 \text{ given} &
\end{aligned}
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$\boldsymbol{\xi}_t = (\boldsymbol{c}_t, \boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)_{t\in[T]}$ is assumed to be stagewise independent.

We set $V_{T+1} \equiv 0$ and:

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V_t(x_{t-1}) := \mathbb{E}\left[\begin{array}{ll} \min\limits_{x_t\in\mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & \boldsymbol{A}_t x_t + \boldsymbol{B}_t x_{t-1} \leqslant \boldsymbol{b}_t \end{array}\right]
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Quantization of a MSLP

The distribution of $(\boldsymbol{c}_t, \boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)_{t\in[T]}$ is often discretized

$$V_t(x_{t-1}) \simeq V_t^d(x_{t-1}) := \sum_{k=1}^K p_k \underbrace{\begin{aligned} \min_{x_t\in\mathbb{R}^{n_t}} \quad & c_{t,k}^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} \quad & A_{t,k}x_t + B_{t,k}x_{t-1} \leqslant b_{t,k} \end{aligned}}_{\widetilde{V}_t(x_{t-1},\xi_{t,k})}$$

Scenario drawn by Monte Carlo : Sample Average Approximation

Two-stage case:

$$\min_{x\in X} c^\top x + V_N^{SAA}(x) \quad \text{where} \quad V_N^{SAA}(x) := \frac{1}{N}\sum_{k=1}^N \widetilde{V}_t(x,\xi^k) \quad (2SLP_N)$$

By statistical results, $Val(2SLP_N) \to_{N\to\infty} Val(2SLP)$.

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Exact quantization

Definition

We say that an MSLP admits an exact quantization if there exists a finitely supported $(\check{\boldsymbol{c}}_t, \check{\boldsymbol{A}}_t, \check{\boldsymbol{B}}_t, \check{\boldsymbol{b}}_t)_{t \in [T]}$ that yields the same expected cost-to-go functions, $(V_t)_{t \in [T]}$.
➡ the MSLP is equivalent to a problem on a finite scenario tree.

Questions:

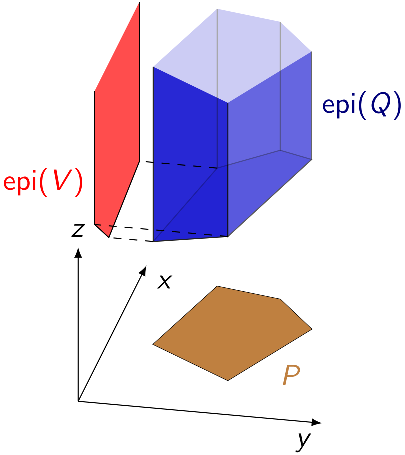
1. Under which condition does there exist an exact quantization ?
2. Can we construct a (uniform) exact quantization ?
3. How does the quantization procedure depends on the noise's law ?

Exact quantization and polyhedrality

- We consider

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \boldsymbol{c}^\top y + V_{t+1}(y) \\ \text{s.t.} & \boldsymbol{B}x + \boldsymbol{A}y \leqslant \boldsymbol{b} \end{array}\right]$$

➥ Assume $V_{t+1} \equiv 0$ for now[1]



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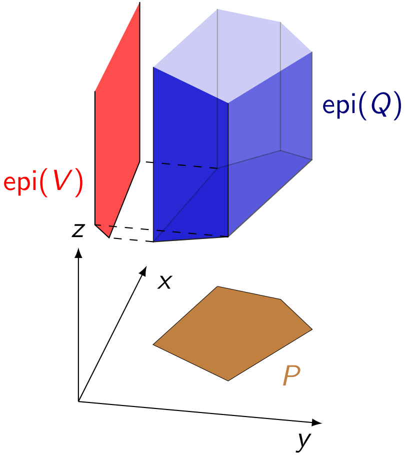

- If the problem is deterministic, then V is polyhedral by projection of the coupling polyhedron
- If the noise is finitely supported, then V is polyhedral

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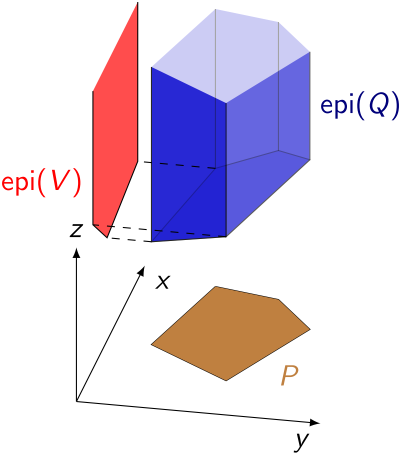

➥ Existence of exact quantization imply polyhedrality of V.

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Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x - y \leqslant 0 \\ & y \geqslant 1 \end{array}\right]$$

$$= \mathbb{E}\big[\max(\boldsymbol{u}x, 1)\big]$$

$$= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

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Remaining case: only $\boldsymbol{c}$ stochastic

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & \boldsymbol{c}^\top y \\ \text{s.t.} & Bx + Ay \leqslant h \end{bmatrix} = \mathbb{E}\big[\min_{y\in\mathbb{R}^m}\big(\boldsymbol{c}^\top y + \mathbb{I}_{Bx+Ay\leqslant h}\big)\big]$$

Theorem (FGL 2021)

If A, B and b are deterministic, then for all distributions of $\boldsymbol{c}$ such that V is well defined, there exists an exact quantization (and V is polyhedral).

➥ This extends easily to finitely supported random $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{b}$.

Let's dive in !

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Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \; y_1 \leqslant x, \; y_2 \leqslant x\}$$

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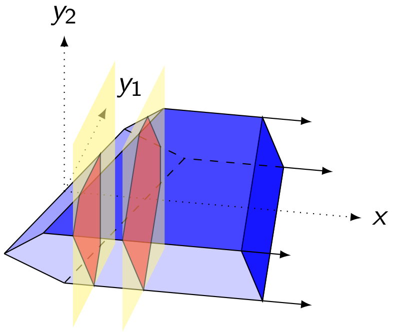


Normal fan $\mathcal{N}(P_x)$

Definition

The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

with $N_{P_x}(y) = \{c \mid \forall y' \in P_x,\ c^\top(y' - y) \leqslant 0\}$ *the normal cone of P_x at y.*

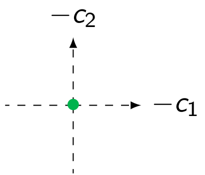
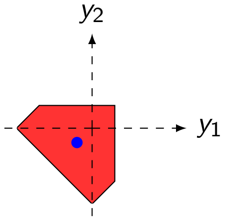


Figure: $N_{P_x}(y)$ for $x = 0.3$



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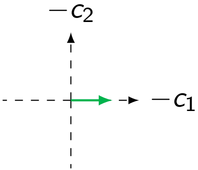

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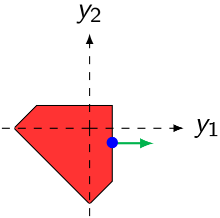

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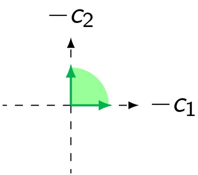

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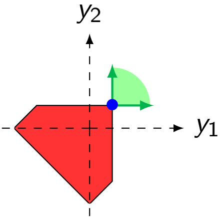

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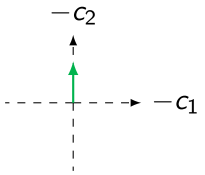

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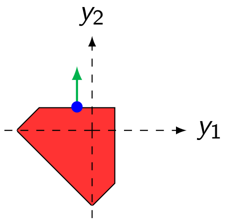

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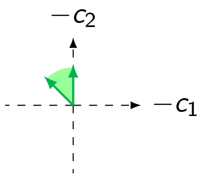

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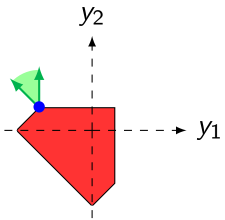

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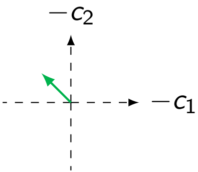

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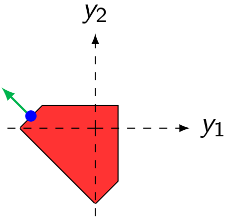

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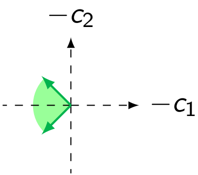

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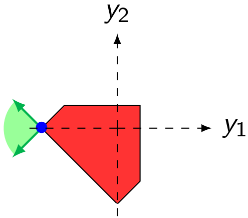

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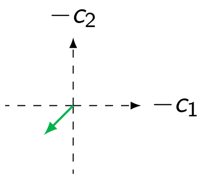

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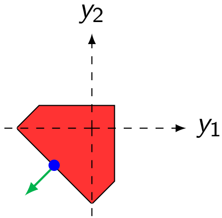

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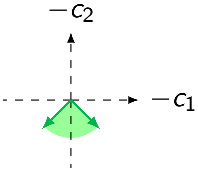

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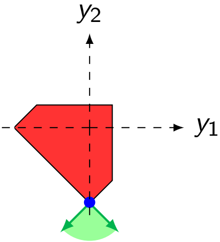

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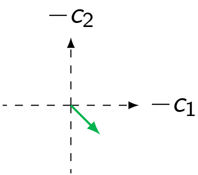

Figure: $N_{P_x}(y)$ for $x = 0.3$

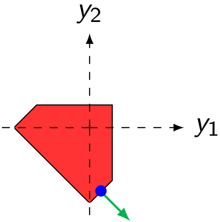

Figure: P_x, y and $N_{P_x}(y)$ for $x = 0.3$

Normal fan $\mathcal{N}(P_x)$

Definition

The normal fan of the fiber P_x is

$$\mathcal{N}(P_x) := \{N_{P_x}(y) \mid y \in P_x\}$$

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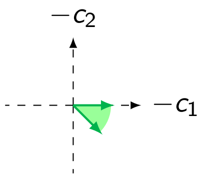

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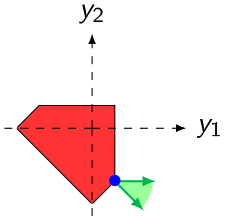

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Proposition

If P_x is bounded, $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ *is a partition of $\mathbb{R}^m$.*

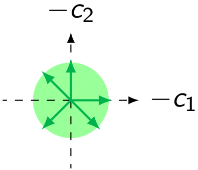


Figure: $\mathcal{N}(P_x)$ for $x = 0.3$

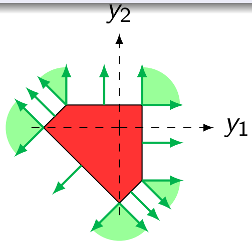


Figure: P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

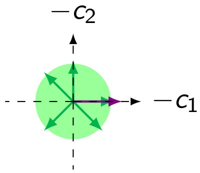
$\mathcal{N}(P_x)$: partition of cost coherent with the min

For a given x, we have

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$

For any $N \in \mathcal{N}(P_x)$, $-c \mapsto \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{y \in P_x} c^\top y$ is a face of P_x.



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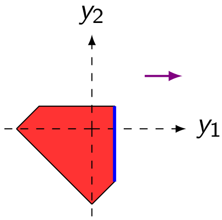

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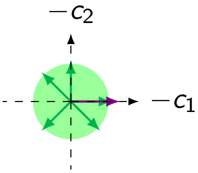
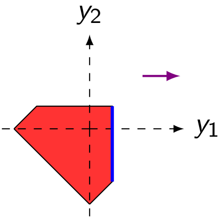


Figure: Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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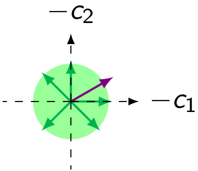
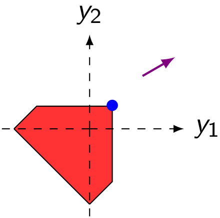


Figure: Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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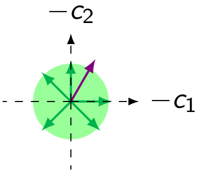

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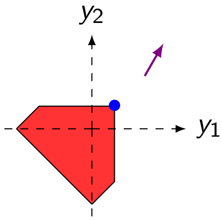

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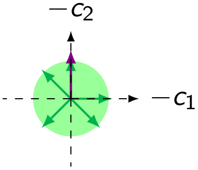

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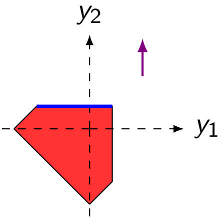

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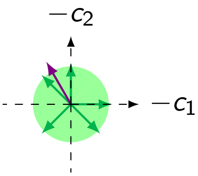

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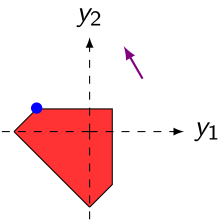

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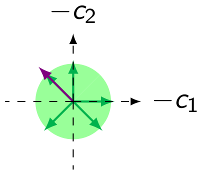

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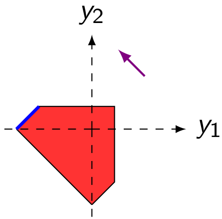

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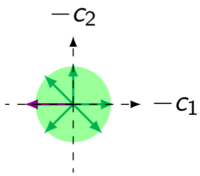

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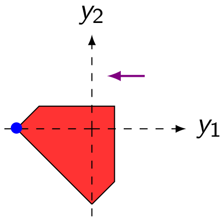

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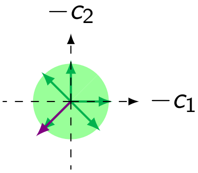

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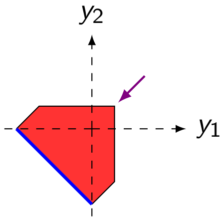

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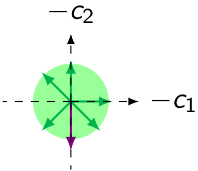

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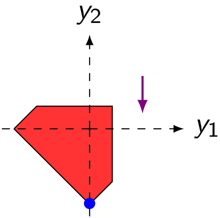

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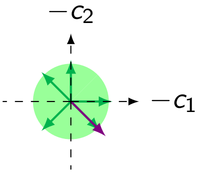

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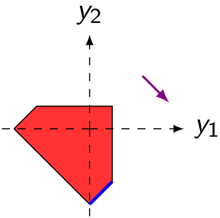

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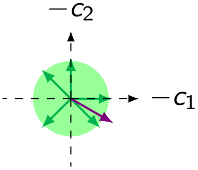

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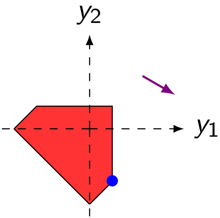

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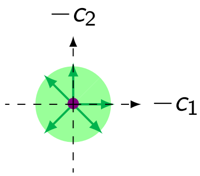

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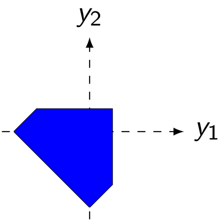

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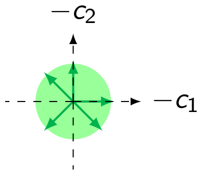

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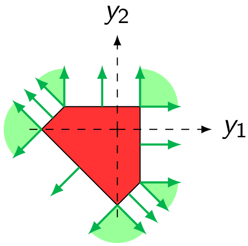

Figure: P_x for $x = 0.3$

General cost $\boldsymbol{c}$ is equivalent to discrete cost $\check{\boldsymbol{c}}$ for given x

For a given x,

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\boldsymbol{c}^\top \mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N}\big] y_N(x)$$

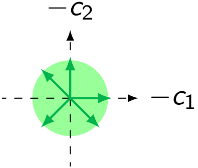

Figure: $\mathcal{N}(P_x)$ for $x = 0.3$

We draw a continuous cost $\boldsymbol{c}$.

General cost $\boldsymbol{c}$ is equivalent to discrete cost $\check{\boldsymbol{c}}$ for given x

For a given x,

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where $y_N \in \arg\min_y \underbrace{c^\top}_{\in -\operatorname{ri} N} y$.

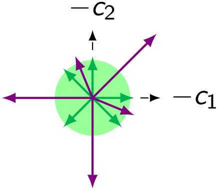

Figure: $\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$p_N := \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

Instead of drawing a general $\boldsymbol{c}$, we draw a discrete cost $\check{\boldsymbol{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

General cost $\boldsymbol{c}$ is equivalent to discrete cost $\check{\boldsymbol{c}}$ for given x

For a given x,

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big]$$
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\boldsymbol{c}^\top \mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N}\big] y_N(x)$$
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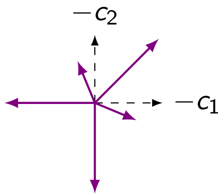

Figure: $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

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$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

Instead of drawing a general $\boldsymbol{c}$, we draw a discrete cost $\check{\boldsymbol{c}}$ indexed by the finite collection $\mathcal{N}(P_x)$.

Contents

$x \mapsto \mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Bx + Ay \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Bx + Ay \leqslant b\}$$

$$x = -0.4$$

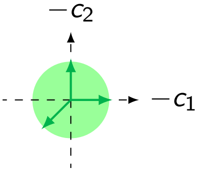


Figure: $\mathcal{N}(P_x)$

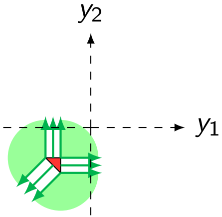


Figure: P_x and $\mathcal{N}(P_x)$

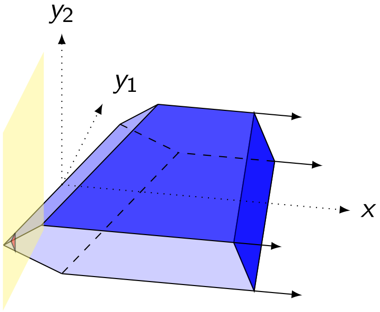


Figure: P and P_x

$x \mapsto \mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Bx + Ay \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Bx + Ay \leqslant b\}$$

$$x = -0.3$$

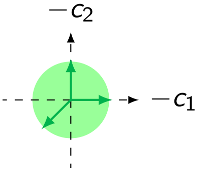
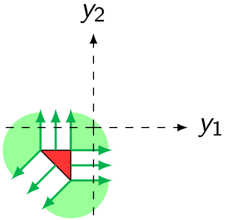


Figure: $\mathcal{N}(P_x)$



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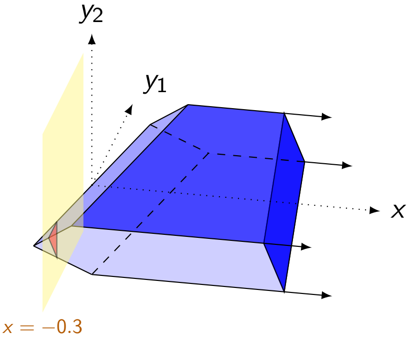

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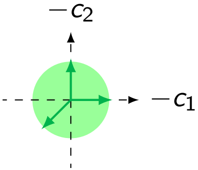
$$x = -0.2$$



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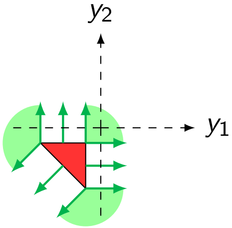

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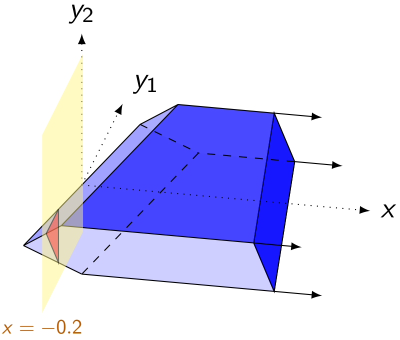

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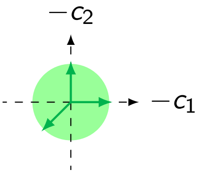

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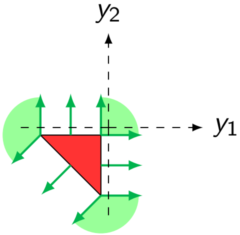

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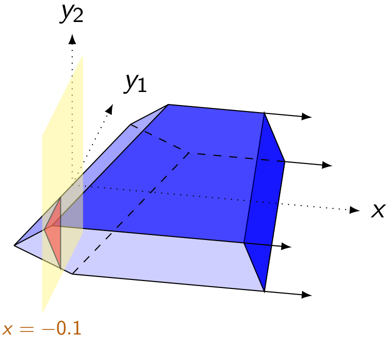

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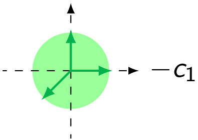
$x = 0$



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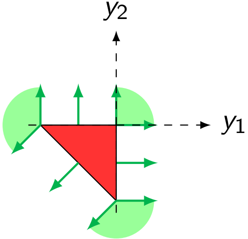

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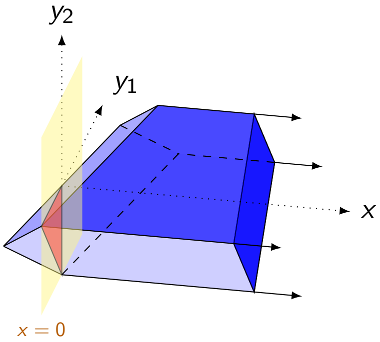

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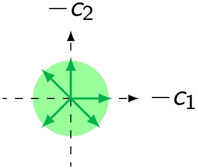

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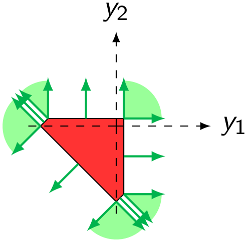

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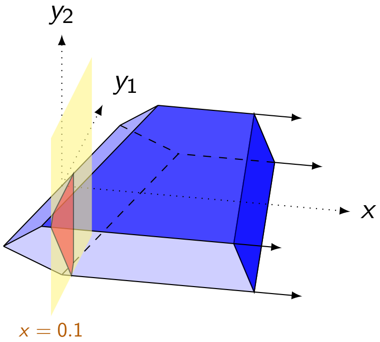

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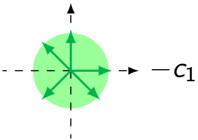


Figure: $\mathcal{N}(P_x)$

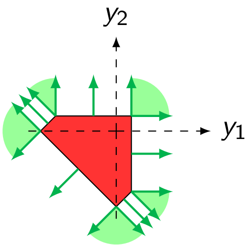
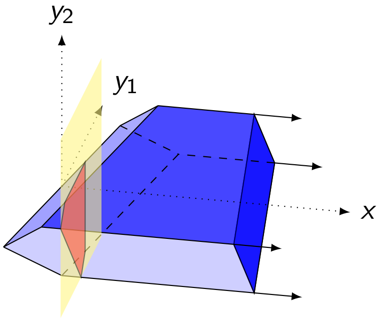


Figure: P_x and $\mathcal{N}(P_x)$



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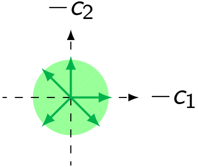


Figure: $\mathcal{N}(P_x)$

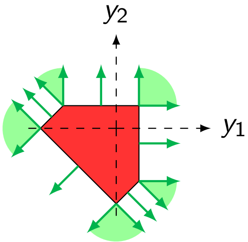
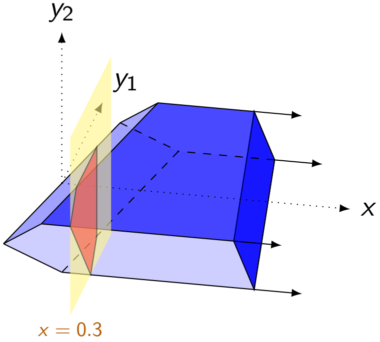


Figure: P_x and $\mathcal{N}(P_x)$



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$x = 0.4$

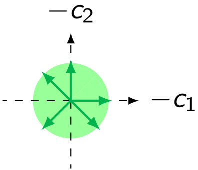


Figure: $\mathcal{N}(P_x)$

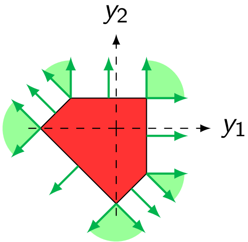
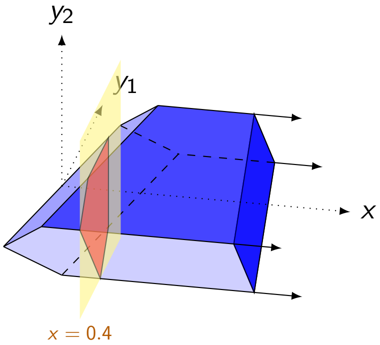


Figure: P_x and $\mathcal{N}(P_x)$



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$x \mapsto \mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Bx + Ay \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Bx + Ay \leqslant b\}$$

$$x = 0.5$$

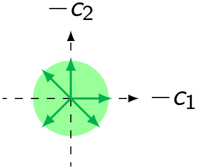


Figure: $\mathcal{N}(P_x)$

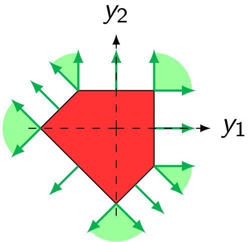
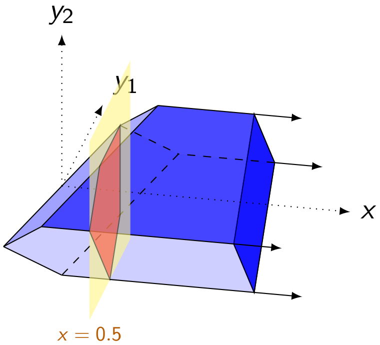


Figure: P_x and $\mathcal{N}(P_x)$



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$x \mapsto \mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Bx + Ay \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Bx + Ay \leqslant b\}$$

$$x = 0.6$$

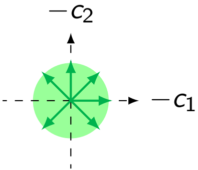


Figure: $\mathcal{N}(P_x)$

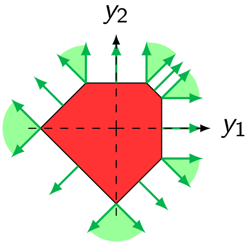
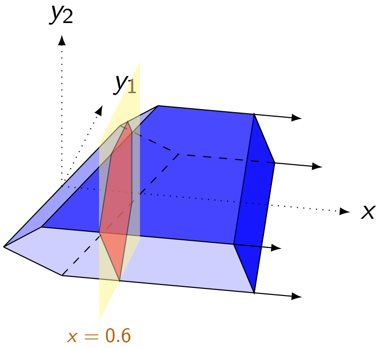


Figure: P_x and $\mathcal{N}(P_x)$



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$x \mapsto \mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Bx + Ay \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Bx + Ay \leqslant b\}$$

$$x = 0.7$$

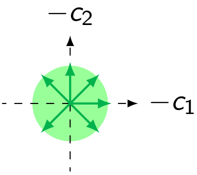


Figure: $\mathcal{N}(P_x)$

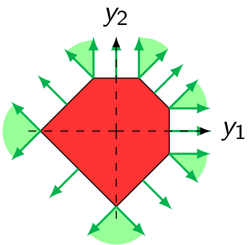
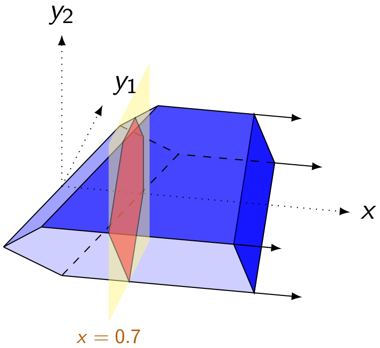


Figure: P_x and $\mathcal{N}(P_x)$



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$$x = 0.8$$

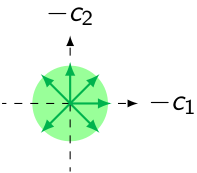
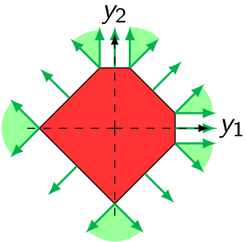


Figure: $\mathcal{N}(P_x)$



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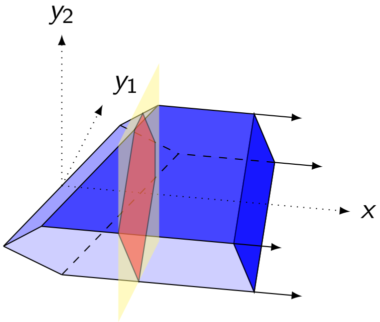

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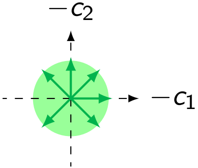
$x = 0.9$



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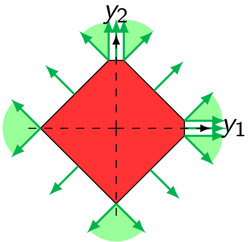

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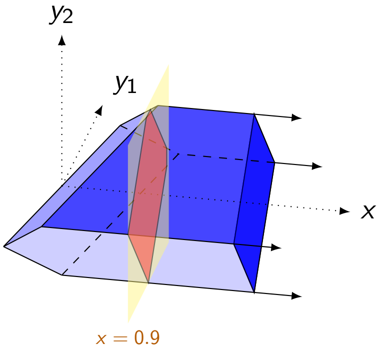

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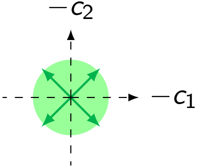
$x = 1$



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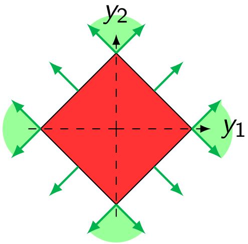

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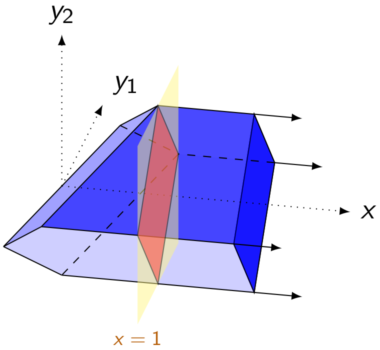

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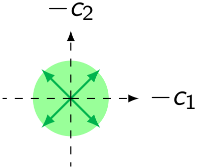
$$x = 1.1$$



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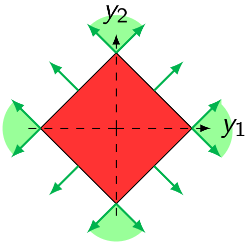

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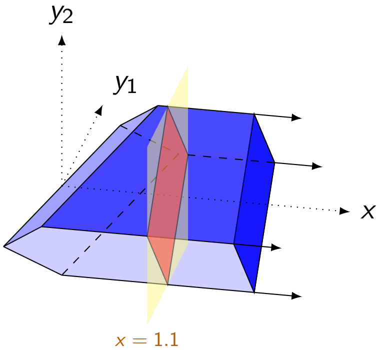

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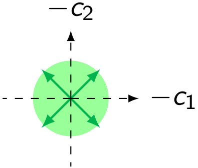

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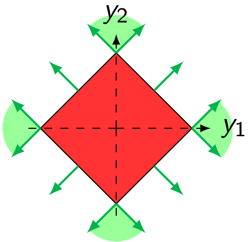

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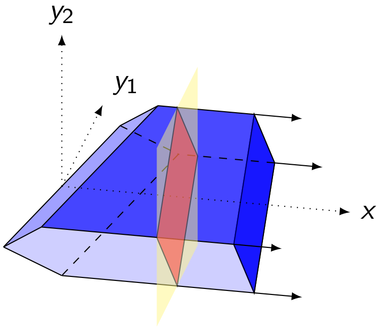

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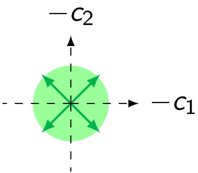

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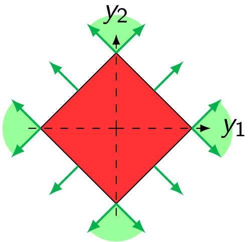

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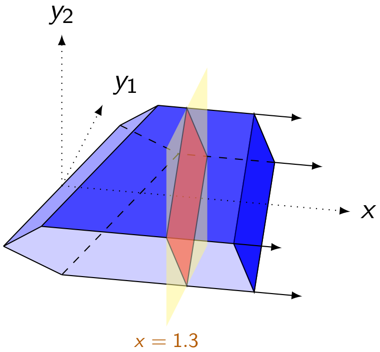

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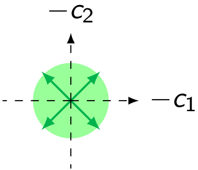

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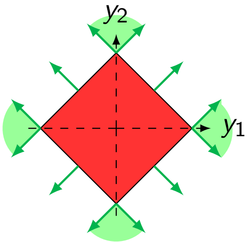

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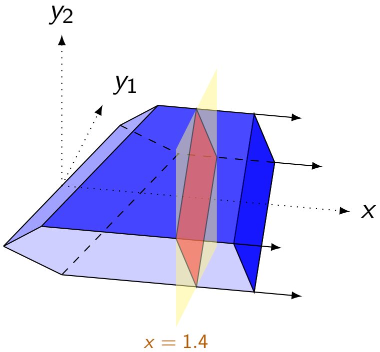

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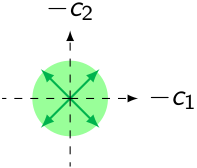


Figure: $\mathcal{N}(P_x)$

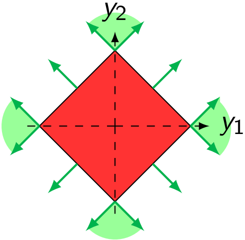


Figure: P_x and $\mathcal{N}(P_x)$

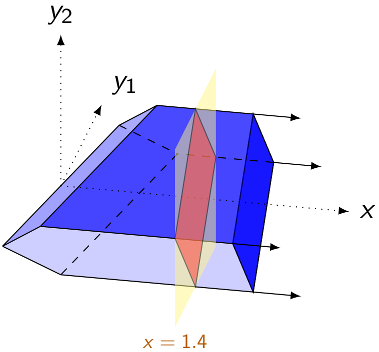
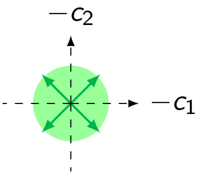


Figure: P and P_x

$x \mapsto \mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Bx + Ay \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Bx + Ay \leqslant b\}$$

$$x = 1.4$$



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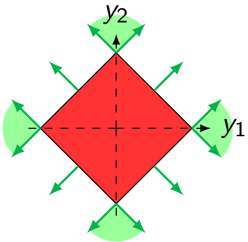

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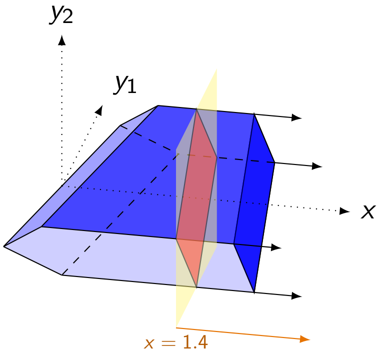

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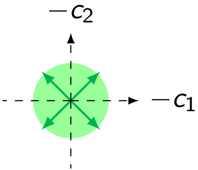

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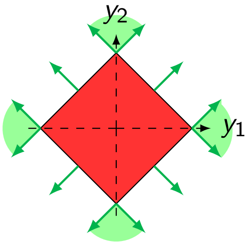

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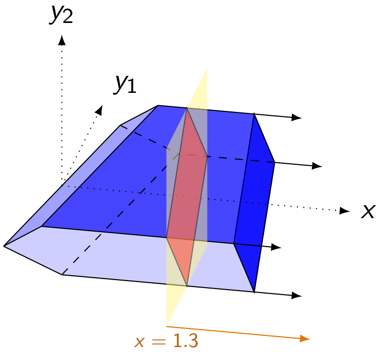

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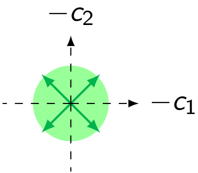
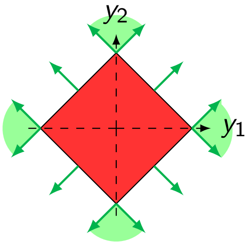


Figure: $\mathcal{N}(P_x)$



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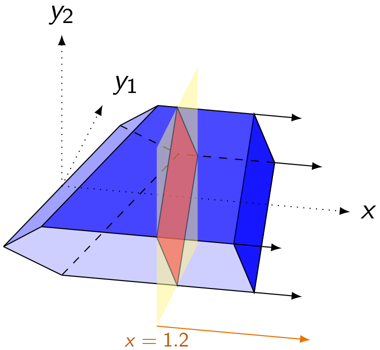

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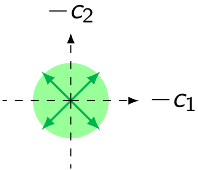
$x = 1.1$



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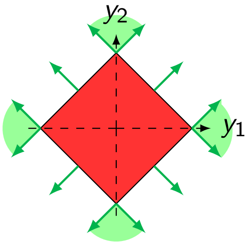

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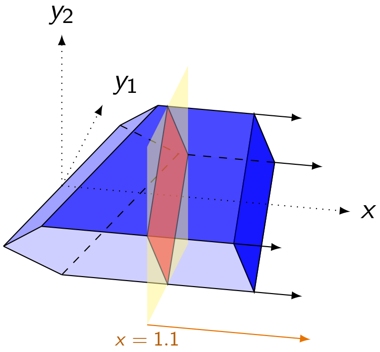

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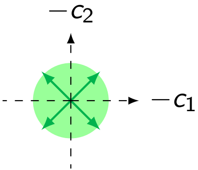
$$x = 1$$



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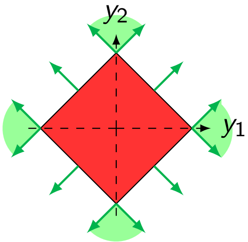

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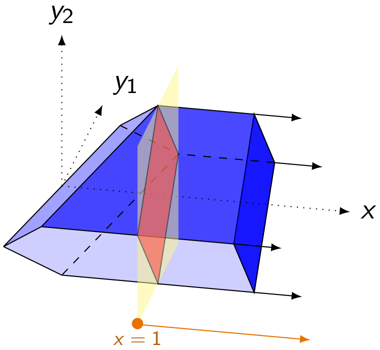

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$x = 0.9$

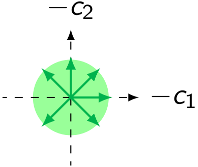
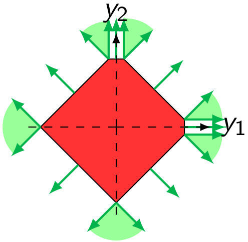


Figure: $\mathcal{N}(P_x)$



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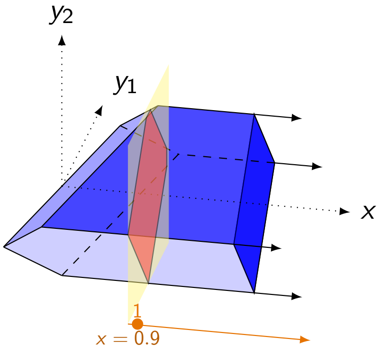

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$x = 0.8$

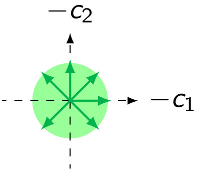


Figure: $\mathcal{N}(P_x)$

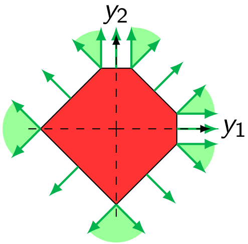
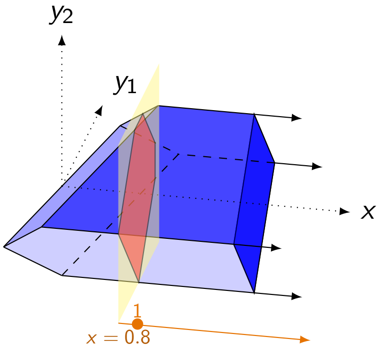


Figure: P_x and $\mathcal{N}(P_x)$



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$x = 0.7$

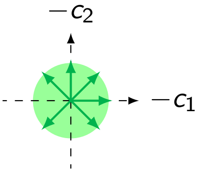


Figure: $\mathcal{N}(P_x)$

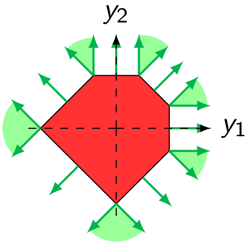
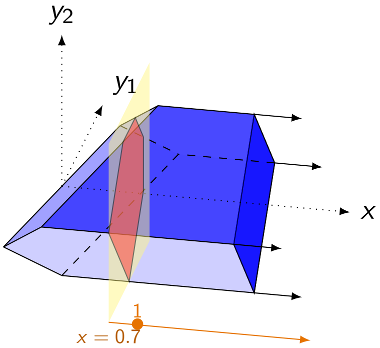


Figure: P_x and $\mathcal{N}(P_x)$



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$x = 0.6$

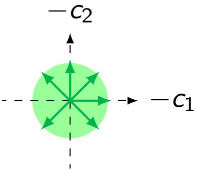
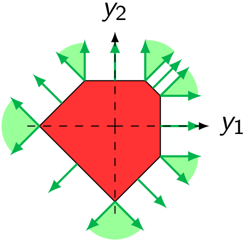


Figure: $\mathcal{N}(P_x)$



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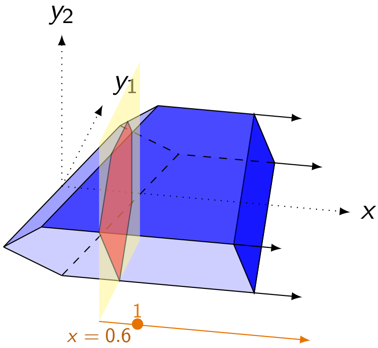

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$$x = 0.5$$

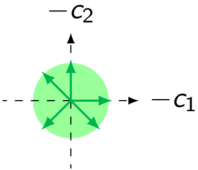


Figure: $\mathcal{N}(P_x)$

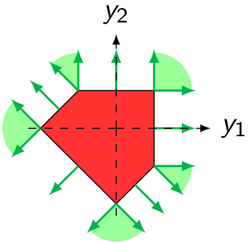
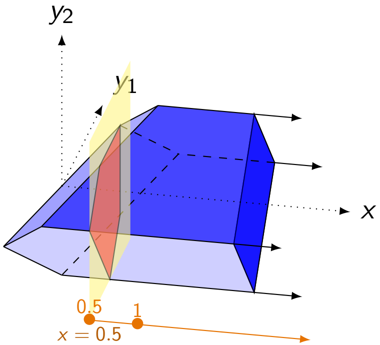


Figure: P_x and $\mathcal{N}(P_x)$



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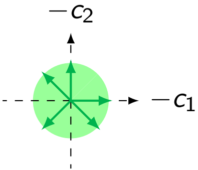
$x = 0.4$



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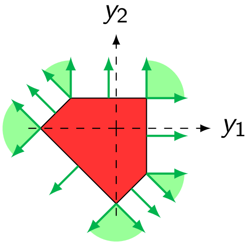

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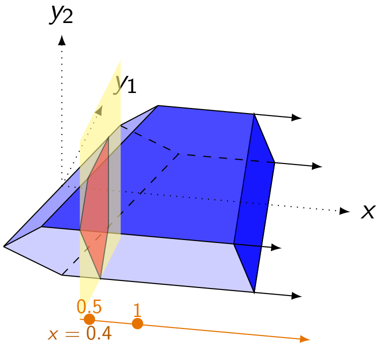

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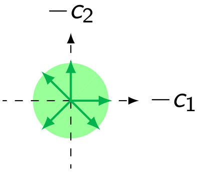
$$x = 0.3$$



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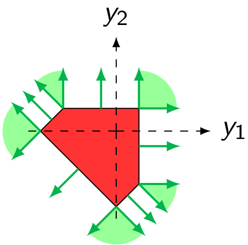

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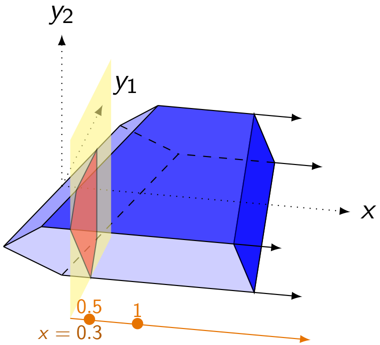

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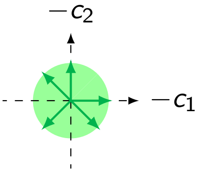

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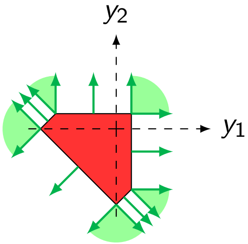

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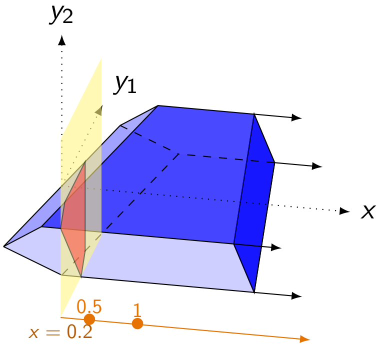

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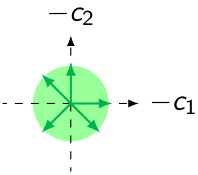

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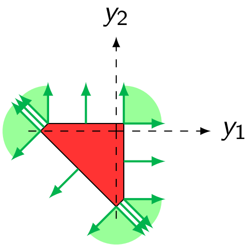

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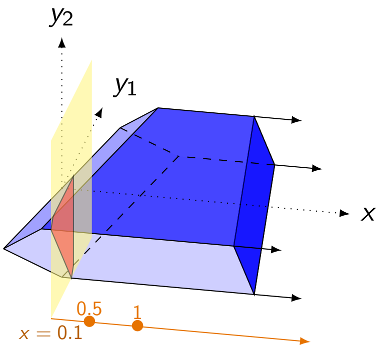

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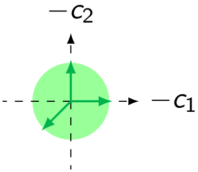
$x = 0$



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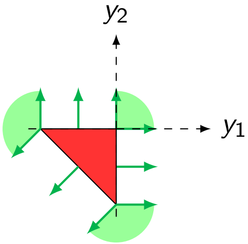

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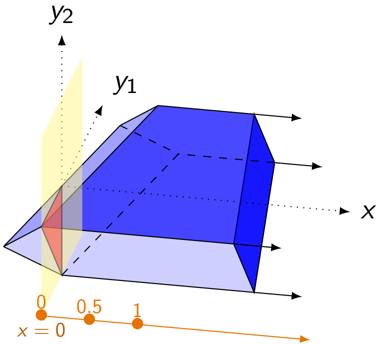

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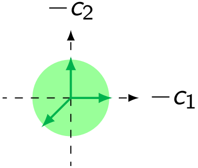
$$x = -0.1$$



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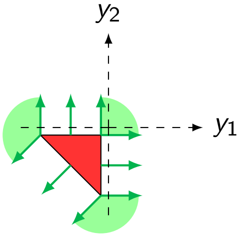

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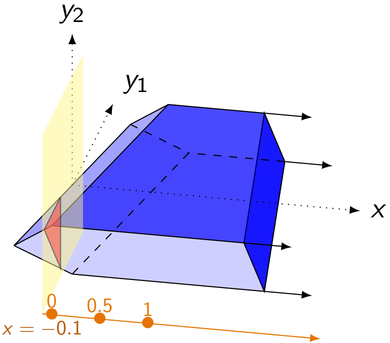

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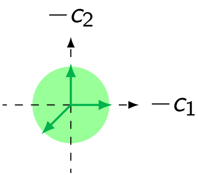

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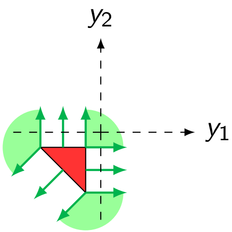

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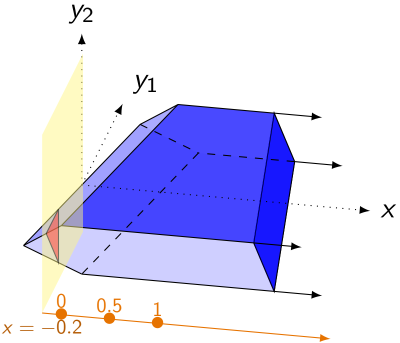

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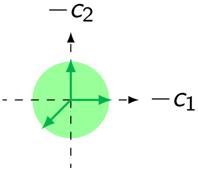
$$x = -0.3$$



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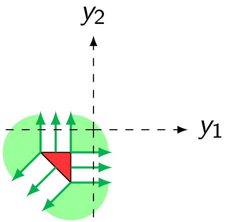

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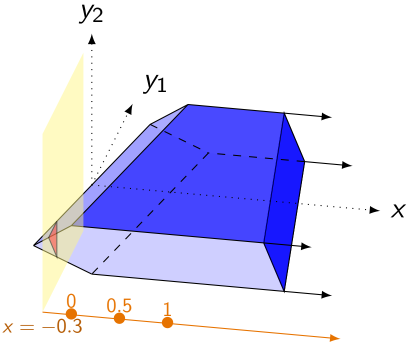

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$$P := \{(x, y) \mid Bx + Ay \leqslant b\} \quad \text{and} \quad P_x := \{y \mid Bx + Ay \leqslant b\}$$

$$x = -0.4$$

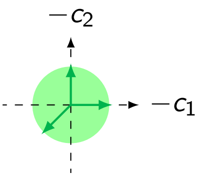


Figure: $\mathcal{N}(P_x)$

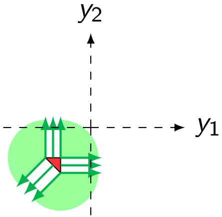


Figure: P_x and $\mathcal{N}(P_x)$

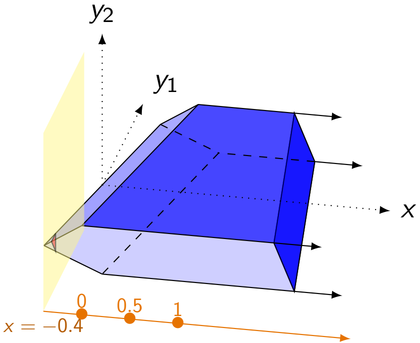


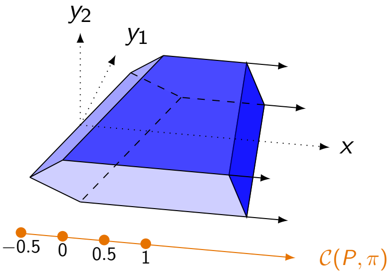
Figure: P and P_x

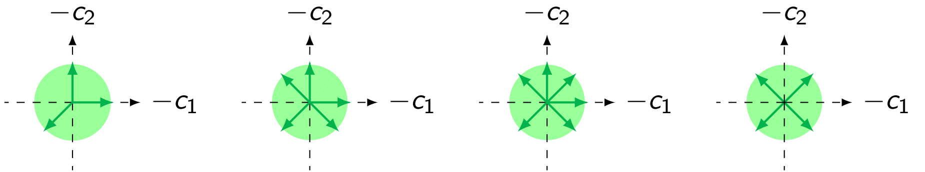
What are the constant regions of $x \mapsto \mathcal{N}(P_x)$?

Lemma (general knowledge[1])

There exists a collection $\mathcal{C}(P, \pi)$ called the **chamber complex** *whose relative interior of cells are the constant regions of $x \mapsto \mathcal{N}(P_x)$.*

I.e, for $\sigma \in \mathcal{C}(P, \pi)$ and $x, x' \in \mathrm{ri}(\sigma)$, we have $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\quad$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\quad$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\quad$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

[1] sort of

Chamber complex

V is affine on the chamber complex, how is it defined ?

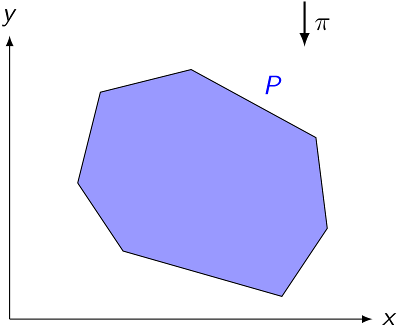
Definition (Billera, Sturmfels 92)

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \mapsto x$

$$\pi(E) := \{x \in \mathbb{R}^n \quad | \quad \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

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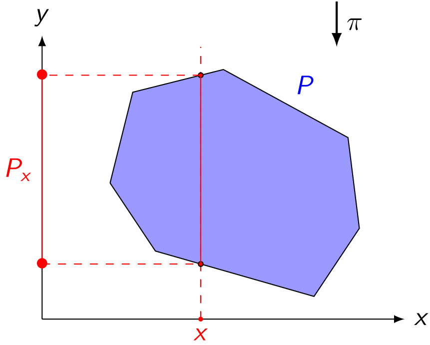

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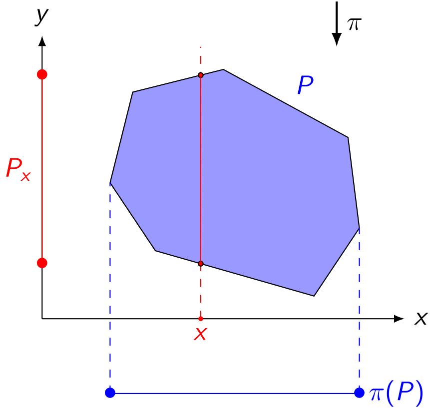

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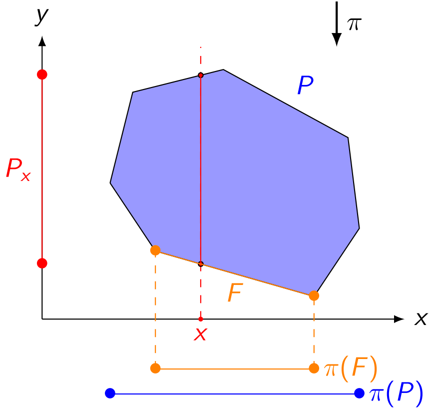

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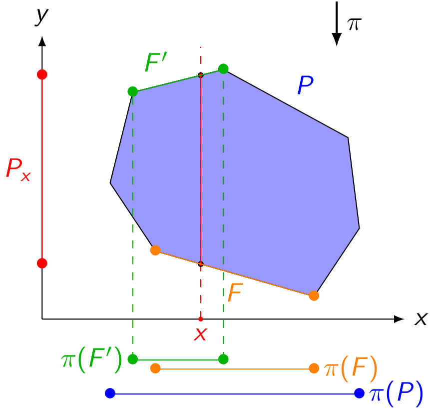

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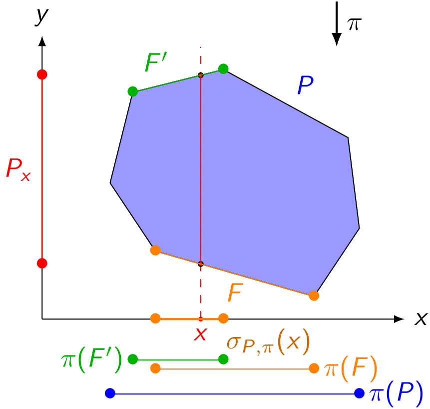

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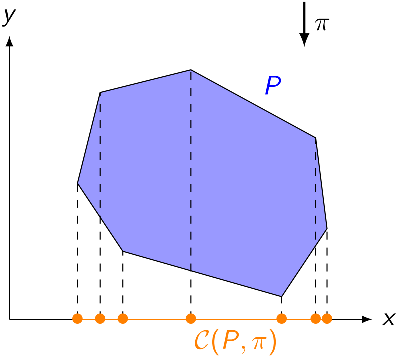

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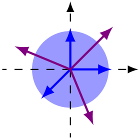
Common Refinement of Normal Fans

We can quantize $\boldsymbol{c}$ on each chamber.



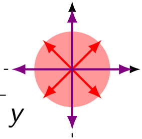
$\mathcal{N}_\sigma$ and $\check{\boldsymbol{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

For all $x' \in \mathrm{ri}(\tau)$,

$$V(x') = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$



$\mathcal{N}_\tau$ and $\check{\boldsymbol{c}}$

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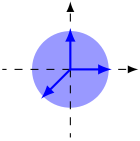

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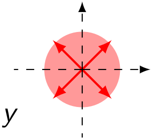
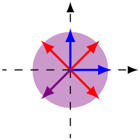


$\mathcal{N}_\tau$

We take the *common refinement*:

$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



$\mathcal{N}_\sigma \wedge \mathcal{N}_\tau$

For all $x \in \mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma \wedge \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$

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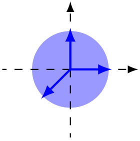


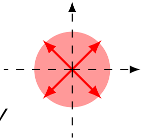
$\color{blue}{\mathcal{N}_\sigma}$

For all $x \in \mathrm{ri}(\color{blue}{\sigma})$,

$$V(x) = \sum_{N \in \color{blue}{\mathcal{N}_\sigma}} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x' \in \mathrm{ri}(\color{red}{\tau})$,

$$V(x') = \sum_{N \in \color{red}{\mathcal{N}_\tau}} p_N \min_{y \in P_x} \check{c}_N^\top y$$



$\color{red}{\mathcal{N}_\tau}$

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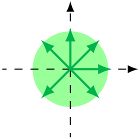

$\color{green}{\mathcal{R}}$

For all $x \in \mathrm{ri}(\color{blue}{\sigma}) \cup \mathrm{ri}(\color{red}{\tau})$,

$$V(x) = \sum_{N \in \color{green}{\mathcal{R}}} p_N \min_{y \in P_x} \check{c}_N^\top y$$

General cost $\boldsymbol{c}$ is equivalent to discrete cost $\check{\boldsymbol{c}}$ for all x

Let's sum up:

1. We had an exact quantization, for given x, on $\mathcal{N}_x$;
2. we can have an exact quantization for x and x' by taking the refinement,
3. we have shown that $x \mapsto \mathcal{N}(P_x)$ is constant on each $\sigma \in \mathcal{C}(P, \pi)$

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Theorem (Uniform quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \,|\, \boldsymbol{c} \in \mathrm{ri}(R)\big]$

Moreover, for all distributions of $\boldsymbol{c}$,
V *is affine on each cell of the chamber complex* $\mathcal{C}(P, \pi)$.

Extension to multistage and stochastic constraints

Theorem

All results generalizes to multistage problem with finitely supported stochastic constraints.

- *The regions where $(V_t)_t$ is affine do not depend on the $(\boldsymbol{c}_t)_t$*
- *We have an exact discretization method that only requires an oracle returning, for any polyhedral cone C, $\mathbb{P}(\boldsymbol{c}_t \in C)$ and $\mathbb{E}\big[\boldsymbol{c}_t \mid \boldsymbol{c}_t \in C\big]$.*

Core idea of the proof :
Iterated chamber complexes

$$\mathcal{P}_{t,\xi} := \mathcal{C}\big((\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t}\big)$$

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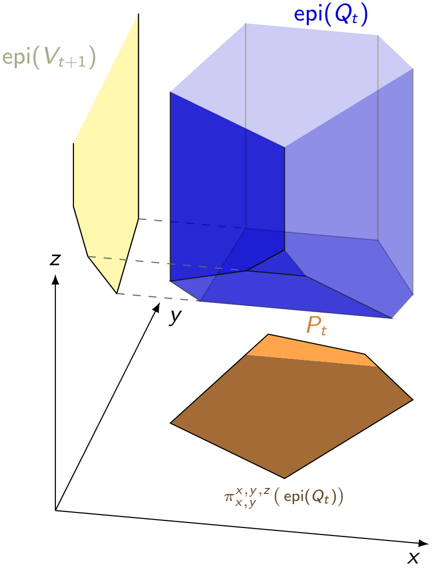
Obtaining a multistage uniform exact quantization

$$V_t(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{x_t \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \\ \text{s.t.} & (x, y) \in P_t \end{array}\right]$$

with $Q_t(x,y) := V_{t+1}(y) + \mathbb{I}_{(x,y)\in P_t}$.



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$$V_t(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{\substack{x_t \in \mathbb{R}^{n_t} \\ z \in \mathbb{R}}} & \boldsymbol{c}_t^\top y + z \\ & \\ \text{s.t.} & (x, y, z) \in \text{epi}(Q_t) \end{array}\right]$$

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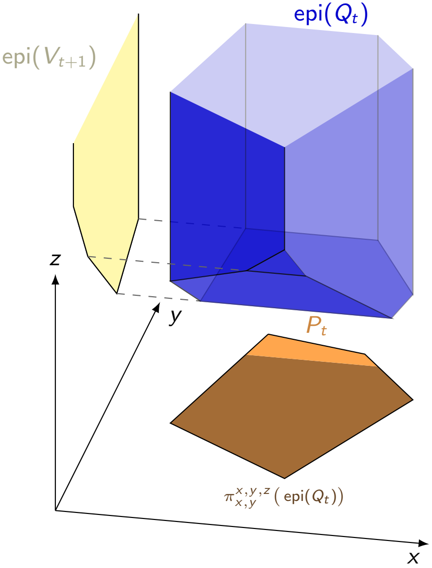

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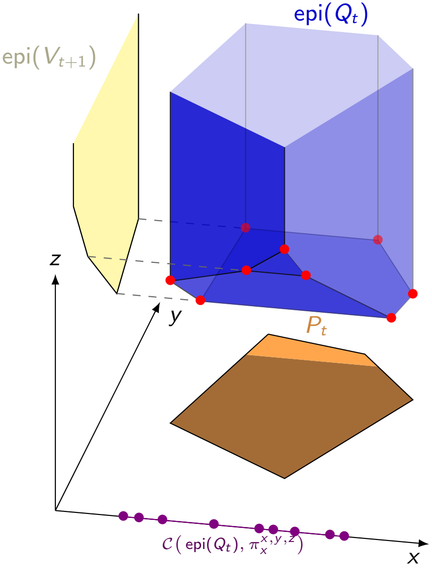

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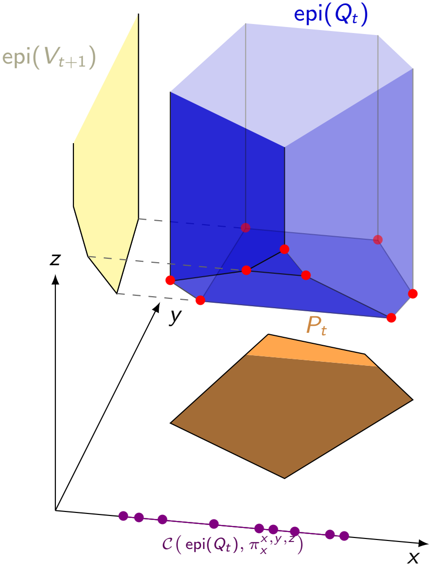

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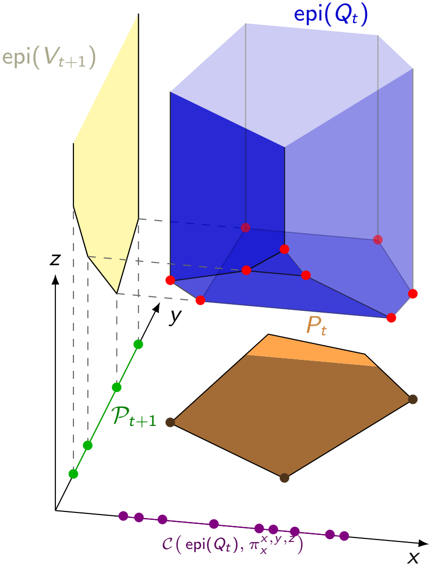

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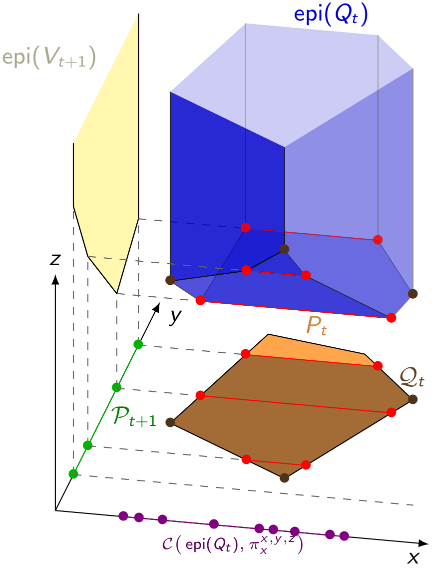

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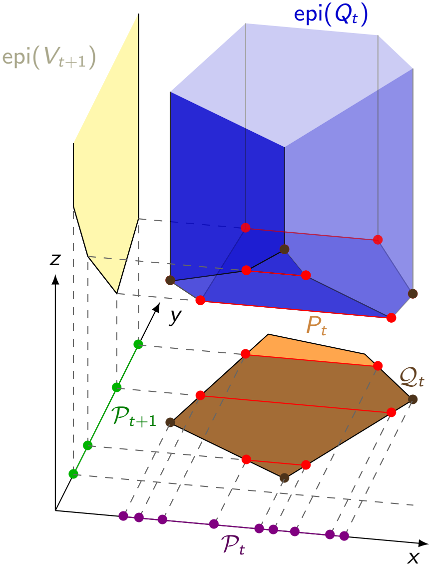

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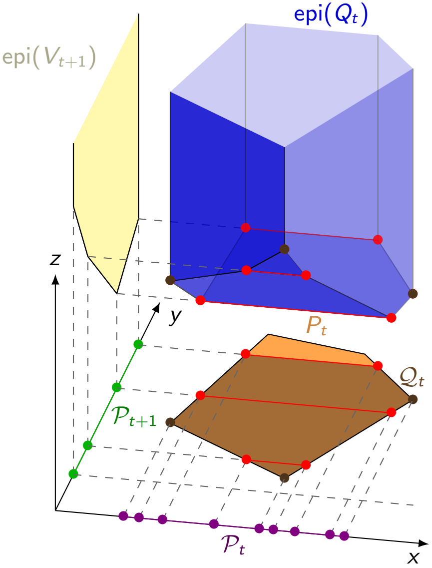
$\mathcal{Q}_t := (\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}(P_t)$

$\mathcal{P}_t := \mathcal{C}(\mathcal{Q}_t, \pi_x^{x,y})$



[FGL21, Lem. 4.1]: $\mathcal{P}_t \preccurlyeq \mathcal{C}(\mathrm{epi}(Q_t), \pi_x^{x,y,z})$

➡ V_t affine on $\mathcal{P}_t$

Contents

Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
$$\mathrm{Vol}\left(\,\mathrm{Conv}(v_1, \cdots, v_n)\right)$$

- $\sharp P$-complete: Dyer and Frieze (1988)
- Polynomial for fixed dimension d: Barvinok (1994)

2-stage linear problem

$$\min_{x \in \mathbb{R}^n} c_0^\top x + \mathbb{I}_{Ax \leqslant b} + \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \boldsymbol{c}^\top y + \mathbb{I}_{Tx + Wy \leqslant h}\Big]$$

- $\sharp P$-hard: Hanasusanto, Kuhn and Wiesemann (2016)
- Polynomial for fixed m ?

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- Polynomial for fixed m: FGL (2021)
  $\rightsquigarrow$ Exact case
  $\rightsquigarrow$ Approximated case

Complexity result multistage

We can generalize to multistage by fixing several dimensions and the horizon.

Theorem (MSLP is polynomial for fixed dimensions)

Assume that n_t, and $\left|\mathrm{supp}(\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)\right|$, for $t = 2, \dots, T$, are fixed integers.[a] *Further, assume that we have an (approximate) oracle taking as argument a cone C and returning in polynomial-time $\mathbb{E}\big[\boldsymbol{c} \in C | (\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t) = (A, B, b)\big]$ and $\mathbb{P}\big(\boldsymbol{c} \in C | (\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t) = (A, B, b)\big)$. Then, MSLP is solvable in polynomial time.*

[a] No requirement for the first decision.

➥ Can be adapted to approximate complexity for a large class of distribution (densities with a bounded total variation).

Contents

Contents

2 stage stochastic linear programming (2SLP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n_+} \quad & c^\top x + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

where $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{h})$ is random whereas q and W are deterministic[2]

$$\begin{aligned} Q(x, \xi) := \min_{y \in \mathbb{R}^m_+} \; & q^\top y & \qquad = \max_{\lambda \in \mathbb{R}^n} \; & (h - Tx)^\top \lambda \\ \text{s.t.} \quad & Tx + Wy = h & \text{s.t.} \quad & W^\top \lambda \leqslant q \end{aligned}$$

We define

$$X := \{x \in \mathbb{R}^n_+ \mid Ax = b\} \qquad D := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

[2] Can be extended to generic random $\boldsymbol{q}$, and finitely supported $\boldsymbol{W}$

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$\rightsquigarrow$ need to discretize $\boldsymbol{\xi}$

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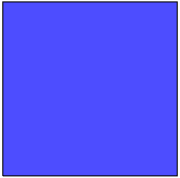
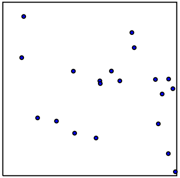
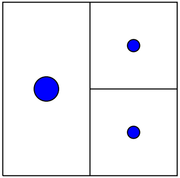
Partitioning the cost-to-go function



ξ continuous



SAA $N = 20$



Partition

$$V(x) = \mathbb{E}\big[Q\big(x, \boldsymbol{\xi}\big)\big] \qquad V_N^{SAA}(x) = \frac{1}{N}\sum_{k=1}^{N} Q\big(x, \xi^k\big) \qquad V_{\mathcal{P}}(x)$$

Definition (Partitioned expected-cost-go)

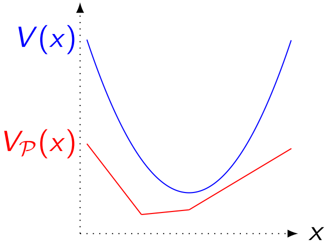
Let $\mathcal{P}$ be a $\mathbb{P}$-partition of Ξ, we define

$$V_{\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \mathbb{P}\big[P\big] Q\big(x, \mathbb{E}\big[\boldsymbol{\xi} | P\big]\big)$$

Properties of partitioned cost-to-go

Recall that

$$V(x) = \mathbb{E}\Big[Q(x,\xi)\Big]$$
$$V_{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big]\, Q\big(x, \mathbb{E}\big[\xi|P\big]\big)$$



- $Q(x,\cdot)$ is convex $\rightsquigarrow$ $V_{\mathcal{P}} \leqslant V$.
- $Q\big(\cdot,\mathbb{E}\big[\xi|P\big]\big)$ is polyhedral $\rightsquigarrow$ $V_{\mathcal{P}}$ is polyhedral.

Finally,

$$\min_{x\in X} c^\top x + V_{\mathcal{P}}(x) \qquad (2SLP_{\mathcal{P}})$$

is equivalent to

$$\min_{x\in X,(y_P)_{P\in\mathcal{P}}} \quad c^\top x + \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] q^\top y_P$$
$$\mathbb{E}\big[\boldsymbol{T}|P\big]x + W y_P \leqslant \mathbb{E}\big[\boldsymbol{h}|P\big] \qquad \forall P\in\mathcal{P}$$

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$$V(x) = \mathbb{E}\Big[Q(x,\boldsymbol{\xi})\Big]$$

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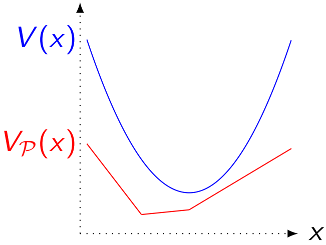

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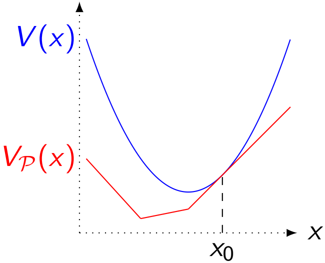
$$\begin{aligned} \min_{x\in X, (y_P)_{P\in\mathcal{P}}} \quad & c^\top x + \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] q^\top y_P \\ & \mathbb{E}\big[\boldsymbol{T}|P\big]x + W y_P \leqslant \mathbb{E}\big[\boldsymbol{h}|P\big] \qquad \forall P \in \mathcal{P} \end{aligned}$$

Adapted partition

Definition

We say that a partition $\mathcal{P}$ is adapted to x_0 if

$$V_{\mathcal{P}}(x_0) = V(x_0) := \mathbb{E}\big[Q(x_0, \xi)\big]$$



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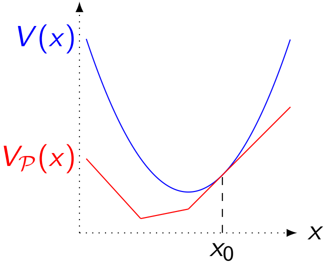

Definition

An partition oracle is a function taking a first stage decision x^k as argument and returning an partition of Ξ.

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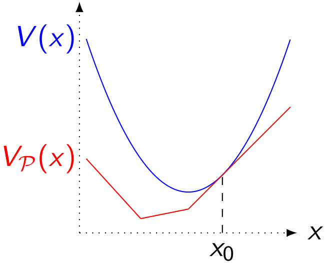

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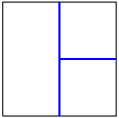
Refinement

$\mathcal{R}$ refines $\mathcal{P}$ ($\mathcal{R} \preccurlyeq \mathcal{P}$) if

$$\forall R \in \mathcal{R}, \exists P \in P, R \subset P$$

[$\mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P}$ if $\mathcal{R}$ refines $\mathcal{P}$ up to $\mathbb{P}$-null sets.]



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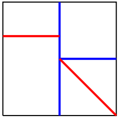

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Then, $\quad \mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P} \Rightarrow V_{\mathcal{R}} \geqslant V_{\mathcal{P}}$

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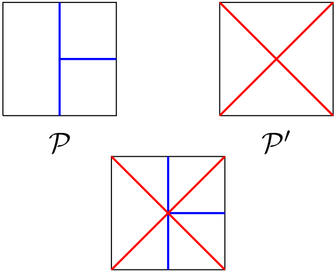
$\mathcal{P}$ $\qquad$ $\mathcal{R}$

The common refinement of $\mathcal{P}$ and $\mathcal{P}'$ is

$$\mathcal{P} \wedge \mathcal{P}' := \{P \cap P' \mid P \in \mathcal{P}, P' \in \mathcal{P}'\}$$

Since $\mathcal{P} \wedge \mathcal{P}'$ refines $\mathcal{P}$ and $\mathcal{P}'$

$$\max(V_{\mathcal{P}}, V_{\mathcal{P}'}) \leqslant V_{\mathcal{P} \wedge \mathcal{P}'}$$



$\mathcal{P}$ $\qquad$ $\mathcal{P}'$ $\qquad$ $\mathcal{P} \wedge \mathcal{P}'$

General framework for APM

$k \leftarrow 0$, $z_U^0 \leftarrow +\infty$, $z_L^0 \leftarrow -\infty$, $\mathcal{P}^0 \leftarrow \{\Xi\}$;
while $z_U^k - z_L^k > \varepsilon$ **do**
  $k \leftarrow k + 1$;
  Solve (for x^k) $\quad z_L^k \leftarrow \min_{x \in X} c^\top x + V_{\mathcal{P}^{k-1}}(x)$;
  $\mathcal{P}_{x^k} \leftarrow \mathrm{Oracle}(x^k)$;
  $\mathcal{P}^k \leftarrow \mathcal{P}^{k-1} \wedge \mathcal{P}_{x^k}$;
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Algorithm 1: Generic framework for APM.

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Algorithm 1: Generic framework for APM.

Theorem (FL2021)

If the oracle is adapted, then x^k is an ε-solution of problem (2SLP) for
$$k \geqslant \left(\frac{L diam(X)}{\varepsilon} + 1\right)^n .$$

Previous APM methods

Lemma (Song & Luedtke)

Let $\mathcal{P}$ a partition of Ξ. $\mathcal{P}$ is adapted at x iff for all set of scenarios $P \in \mathcal{P}$, there exists a common optimal multiplier λ_P, i.e.

$$\forall P \in \mathcal{P}, \quad \exists \lambda_P \in D, \quad \forall \xi_k \in P, \qquad \lambda_P \in \underset{\lambda \in D}{\mathrm{argmax}} (h^k - T^k x)^\top \lambda$$

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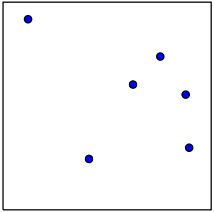
Idea

- Sample a large number of scenario
- without loss of precision aggregate scenarios



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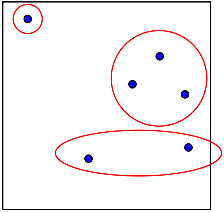

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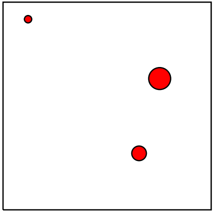

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Lemma (Ramirez-Pico & Moreno)

Let $\mathcal{P}$ a partition of Ξ. If there exists $\lambda(\boldsymbol{\xi})$ such that, for all $P \in \mathcal{P}$,

$$\mathbb{E}\big[\boldsymbol{h}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = \mathbb{E}\big[\boldsymbol{h}^\top \lambda(\boldsymbol{\xi})|P\big]$$
$$x^\top \mathbb{E}\big[\boldsymbol{T}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = x^\top \mathbb{E}\big[\boldsymbol{T}^\top \lambda(\boldsymbol{\xi})|P\big]$$

then $\mathcal{P}$ is an adapted partition.

A (partial) comparison between partition based results

Paper	Song, Luedtke (2015)	Ramirez-Pico, Moreno (2020)	Forcier, L. (2021)
Non-finite supp(ξ)	$\times$	$\checkmark$	$\checkmark$
Explicit oracle	$\checkmark$	$\times$	$\checkmark$
Proof of convergence	$\checkmark$	$\times$	$\checkmark$
Complexity result	$\times$	$\times$	$\checkmark$
Fast iteration	$\checkmark$	$\times$	$\times$

Contents

Local exact quantization and adapted partition

Local exact quantization

random cost

Recall that for a fixed x,

$$\mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leqslant b\}$$

GAPM

random constraints

Similarly, for a given q, and all x,

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ &= \mathbb{E}\Big[\max_{\lambda \in D_q} (\boldsymbol{h} - \boldsymbol{T}x)^\top \lambda\Big] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}{}^\top \lambda \end{aligned}$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$
$$\psi_{N,x} := \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \mid \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$

$$D_q := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(D_q)$ a normal cone of D_q. We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \operatorname{ri} N\}$$

Theorem (FL 2021)

$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$ is an adapted partition to x
i.e. $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x,\boldsymbol{\xi})\big] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\big[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big] \min_{\lambda \in D} \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \,|\, \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\big[\boldsymbol{\xi} \in E_{N,x}\big] Q\Big(\mathbb{E}\big[\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in E_{N,x}\big], x\Big) = V_{\mathcal{R}_x}(x) \end{aligned}$$

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➥ Is it the coarsest one ?

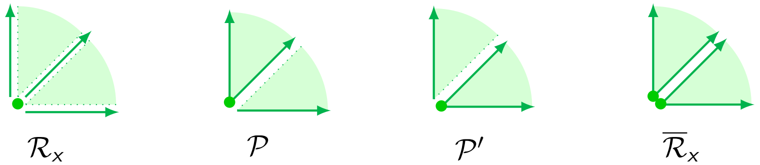
CNS conditions for a partition to be adapted

Theorem (FL 2021)

For $x \in \mathbb{R}^n$ and $\mathcal{P}$ a partition of Ξ, there exists $\overline{\mathcal{R}}_x \succcurlyeq_{\mathbb{P}} \mathcal{R}_x$ such that

$$\mathcal{P} \preccurlyeq_{\mathbb{P}} \overline{\mathcal{R}}_x \iff V_{\mathcal{P}}(x) = V(x).$$

- If $\boldsymbol{\xi}$ admits a density, $\mathcal{R}_x =_{\mathbb{P}} \overline{\mathcal{R}}_x$.
- An oracle is adapted if and only if it returns a partition $\mathcal{P}$ refining $\overline{\mathcal{R}}_x$.



$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \mathrm{ri}(N)\}$$
$$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$$

$$\overline{E}_{N,x} := \{\xi \in \Xi \mid h - Tx \in N\}$$
$$\overline{\mathcal{R}}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)^{\max}\right\}.$$

Stochastic cost and recourse

- We have shown a local exact quantization result for random $\boldsymbol{T}, \boldsymbol{h}$, and deterministic q, W.
- If $\boldsymbol{q}$ and $\boldsymbol{W}$ are finitely supported random variable:
  1. compute an exact quantization $\mathcal{N}_\xi$ for every element of the support;
  2. take the common refinement.

We have seen that we can deal with non-finitely supported $\boldsymbol{q}$ through the chamber complexes.

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Adapted partition for general q

We define coupling constraint and fiber for the dual.

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\begin{aligned}
D_q &:= \{\lambda \in \mathbb{R}^\ell \quad | \quad W^\top \lambda \leqslant q\} \\
\Delta &:= \{(\lambda, q) \in \mathbb{R}^\ell \times \mathbb{R}^m \quad | \quad W^\top \lambda \leqslant q\} \\
\mathcal{R}_{x,q} &:= \{E_{N,x} \quad | \quad N \in \mathcal{N}(D_q)\}
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Recall that $q \mapsto \mathcal{N}(D_q)$ is piecewise constant on $\mathcal{C}(\Delta, \pi_\lambda^{\lambda,q})$ and so is $\mathcal{R}_{x,q}$.
➥ we can take the common refinement of a finite number of $\mathcal{R}_{x,q}$!!

More precisely:

- The chamber complex $\mathcal{C}(\Delta, \pi_\lambda^{\lambda,q}) = \Sigma\text{-fan}(W)$[3].
- For $S \in \Sigma\text{-fan}(W)$ define $\mathcal{R}_{x,S} := \mathcal{R}_{x,q}$ for any $q \in \mathrm{ri}(S)$.

➥ $\{\mathrm{ri}(S) \times R \mid S \in \Sigma\text{-fan}(W), R \in \mathcal{R}_{x,S}\}$ is an adapted partition to x.

[3] The well studied secondary fan of W

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Synthesis of local and uniform quantization results

	$\boldsymbol{W}$	$(\boldsymbol{T}, \boldsymbol{h})$	$\boldsymbol{q}$
Local	$\emptyset$	$\mathcal{R}_x$	$\mathcal{N}(P_x)$
Uniform	$\emptyset$	$\emptyset$	$\bigwedge_{\sigma \in \mathcal{C}(P,\pi)} \mathcal{N}_\sigma$

Contents

Subgradient of partition function

Recall that if $\mathcal{P} \preccurlyeq_{\mathbb{P}} \mathcal{R}_x$ then

$$V_{\mathcal{R}_x}(x) = V_{\mathcal{P}}(x) = V(x)$$

$$V_{\mathcal{R}_x}(\cdot) \leqslant V_{\mathcal{P}}(\cdot) \leqslant V(\cdot)$$

Lemma

Let $x \in \mathrm{dom}(V)$ *and* $\mathcal{P}$ *be a refinement of* $\mathcal{R}_x$*, i.e.* $\mathcal{P} \preccurlyeq \mathcal{R}_x$*, then*

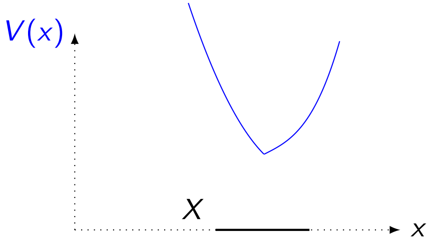
$$\partial V_{\mathcal{R}_x}(x) \subset \partial V_{\mathcal{P}}(x) \subset \partial V(x)$$

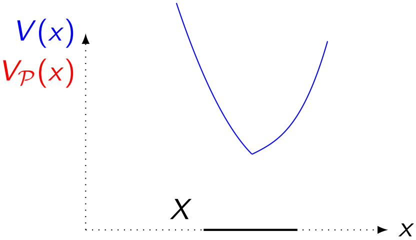
Furthermore, if $x \in \mathrm{ri}\,\mathrm{dom}(V)$*,*

$$\partial V_{\mathcal{R}_x}(x) = \partial V_{\mathcal{P}}(x) = \partial V(x)$$

Link with Benders decomposition and L-shaped

Partition based method can be seen as a tangent cone method: a cutting plane method where we add all active cuts instead of a single one.





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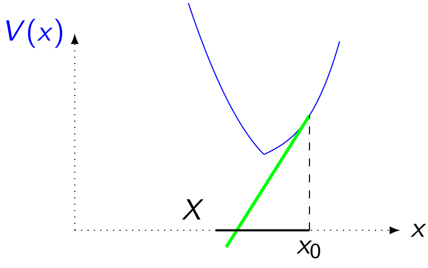

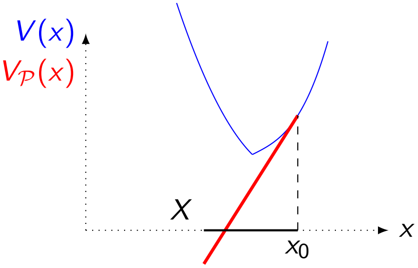

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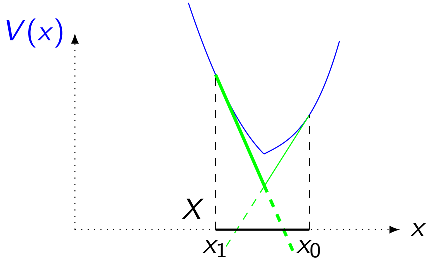

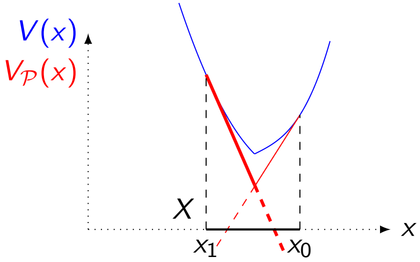

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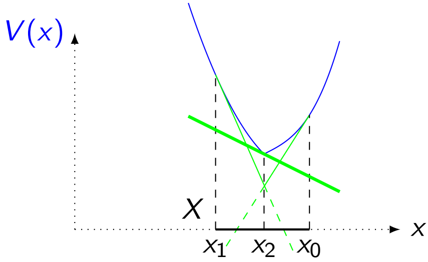

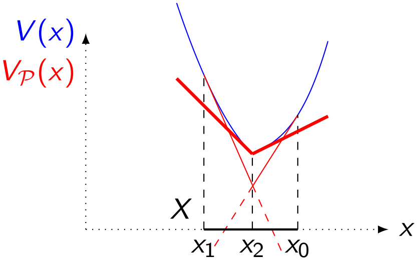

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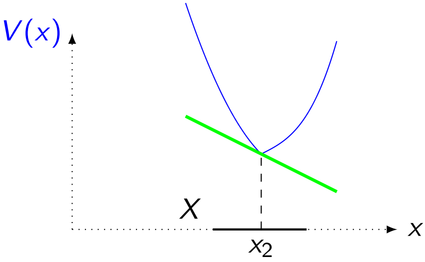

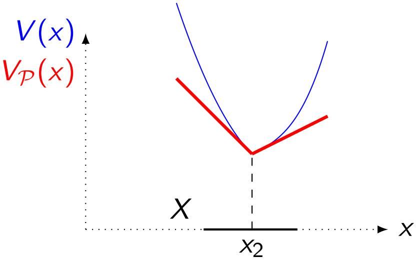

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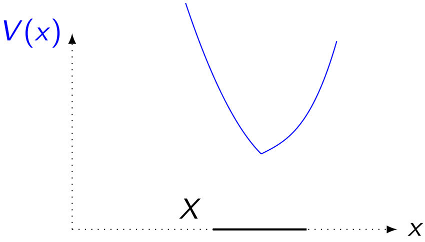

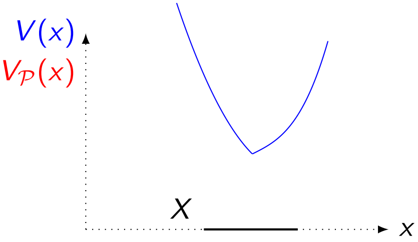

Theorem (Convergence and complexity results)

If $X \cap \mathrm{dom}(V) \subset \mathbb{R}^+$ is contained in a ball of diameter $M \in \mathbb{R}^+$ and $x \to c^\top x + V(x)$ is Lipschitz with constant L then the partition based method finds an ε-solution in at most $\left(\frac{LM}{\varepsilon} + 1\right)^n$ iterations.

Contents

Explicit formulas for usual distributions

Recall that $V_{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] Q\big(x, \mathbb{E}\big[\boldsymbol{\xi}|P\big]\big)$.

Thus, we need to compute $\mathbb{P}\big[C\big]$ and $\mathbb{E}\big[\boldsymbol{\xi} \mid C\big]$ when C is a polyhedron.

Fortunately we have some explicit formulas, valid for S full dimensional simplex or simplicial cone, which can be used through triangulation.

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(\xi)$	$\frac{\mathbb{1}_{\xi\in Q}}{\mathrm{Vol}_d(Q)}\mathcal{L}_{\mathrm{Aff}(Q)}(d\xi)$	$\frac{e^{\theta^\top \xi}\mathbb{1}_{\xi\in K}}{\Phi_K(\theta)}\mathcal{L}_{\mathrm{Aff}(K)}(d\xi)$	$\frac{e^{-\frac{1}{2}\xi^\top M^{-2}\xi}}{(2\pi)^{\frac{m}{2}}\det M}d\xi$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\mathbb{P}\big[S\big]$	$\frac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\frac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)}\prod_{r\in\mathrm{Ray}(S)}\frac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\mathbb{E}\big[\xi \mid S\big]$	$\frac{1}{d}\sum_{v\in\mathrm{Vert}(S)} v$	$\left(\sum_{r\in\mathrm{Ray}(S)}\frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}M\,\mathrm{Ctr}\left(S\cap\mathbb{S}_{m-1}\right)$

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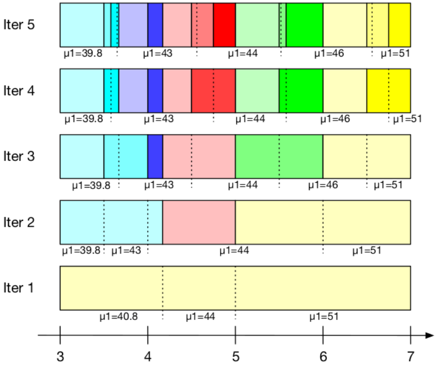
Numerical Results - LandS



Iter	x_1	x_2	x_3	x_4
1	0.833	3.000	4.167	4.000
2	2.500	3.000	3.500	3.000
3	1.833	4.000	3.667	2.500
4	2.000	4.167	3.583	2.250
5	1.917	4.083	3.625	2.375
6	1.875	4.042	3.646	2.438

Iter	LB	UB	Gap
1	378.667	382.711	1.0567%
2	380.122	381.100	0.2567%
3	380.601	380.844	0.0640%
4	380.842	380.893	0.0007%
5	380.843	380.856	0.0004%
6	380.844	380.847	0.0002%

Figure: Results given by GAPM for LandS problem[4]

[4]illustration from Ramirez-Pico and Moreno

Numerical Results - ProdMix

k	x_k	z_L^k	z_U^k	Gap	$\lvert\mathcal{P}_k^{\max}\rvert$
1	$(1333.33, 66.67)$	-18666.67	-16939.71	9.3%	4
2	$(1441.41, 59.57)$	-17873.01	-17383.73	2.7%	9
3	$(1399.05, 57.91)$	-17789.88	-17659.19	0.74%	16
4	$(1379.98, 56.64)$	-17744.67	-17708.00	0.20%	25
5	$(1371.36, 55.71)$	-17718.96	-17709.05	0.056%	36
6	$(1375.55, 56.21)$	-17713.74	-17711.37	0.013%	49

Table: Results for problem Prod-Mix

To compare our approach with SAA, we solved the same problem 100 times, each with 10 000 scenarios randomly drawn, yielding a 95% confidence interval centered in -17711, with radius 2.2.

Conclusions and perspectives

- We have shown how to obtain a (uniform) exact quantization for an MSLP, providing new complexity results. Unfortunately this quantization might be very large.
- We have shown how to use local exact quantization for two-stage problem, in a Benders' like manner.
- Our next steps:
  - Computing and using only local exact quantization in a simplex-like method working on the chamber complexes.
  - Using the APM method for multistage problems, with sampling leading to SDDP methods for non-finitely supported problem.

Y. Song, J. Luedtke
An adaptive partition-based approach for solving two-stage stochastic programs with fixed recourse.
SIAM Journal on Optimization, 25(3), 1344-1367.

C. Ramirez-Pico, E. Moreno
Generalized adaptive partition-based method for two-stage stochastic linear programs with fixed recourse.
Mathematical Programming (2021): 1-20.

M. Forcier, S. Gaubert, V. Leclère
Exact quantization of multistage stochastic linear problems.
arXiv preprint arXiv:2107.09566 (2021).

M. Forcier, V. Leclère
Generalized adaptive partition-based method for two-stage stochastic linear programs: convergence and generalization.
arXiv preprint arXiv:2109.04818 (2021).

M. Forcier, V. Leclère
Convergence of Stochastic Dual Dynamic Programming algorithms for non-finitely supported distributions
soon.

Thank you for listening ! Any question ?