Resilience, Viability and Stochastic Optimization

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Workshop "Robustness and Resilience in Stochastic Optimization and Statistical Learning: Mathematical Foundations"
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May 30, 2022

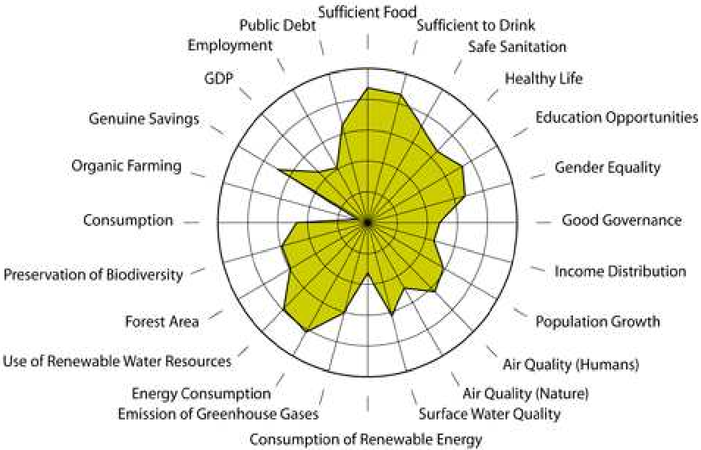
Formal ingredients for an operational definition of resilience

[Holling, 1973] C. S. Holling.
Resilience and stability of ecological systems.
Annual Review of Ecology and Systematics, 4:1–23, 1973.
$\rightarrow$ not a single equation!

Resilience is the capacity of a system to continually change and adapt yet remain within critical thresholds (Stockholm Resilience Centre)

- "continually change", "remain"
  $\rightarrow$ time variable (continuous, discrete)
- "system", "change"
  $\rightarrow$ states, dynamics, dynamical system
- "adapt"
  $\rightarrow$ actions, controls, decisions, strategies, policies, decision rules
- "remain within critical thresholds"
  $\rightarrow$ constraint set, admissibility, viable set, viability

Sustainable Society Index 2010 - World



To make a long story short . . .

Mathematical control theory, viability and stochastic optimization offer material for an operational definition of resilience

Theory. Mathematics provides concepts, tools and methods

- states, controls, uncertainties, dynamics (control theory)
- scenarios, policies, critical thresholds
- (stochastic, robust) viability kernel = viable states
- minimal time of crisis, cost-efficiency (optimization)

Answers. Geometry + Optimization

- Resilient states = viable states
- Measuring resilience as the inverse of the minimal cost (expected, robust) to reach a viability kernel

Tribute to
Jean-Pierre Aubin, Patrick Saint-Pierre, Luc Doyen, Sophie Martin

Our emphasis on the treatment of uncertainties:
stochastic and robust viability, and possible extensions

Outline of the presentation

The viability approach

Handling uncertainty in control theory

The stochastic/robust viability approaches

Measures of resilience and extensions

“Self-promotion, nobody will do it for you” ;-)

Outline of the presentation

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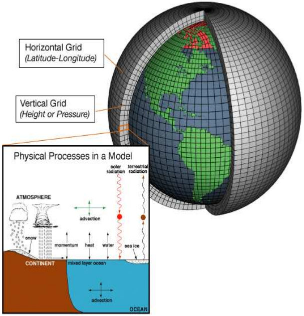
We distinguish two polar classes of models: knowledge models *versus* decision models



Knowledge models:
$1/1\ 000\ 000 \to 1/1\ 000 \to 1/1$
maps

Office of Oceanic and Atmospheric Research (OAR) climate model

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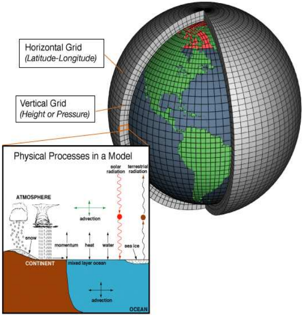
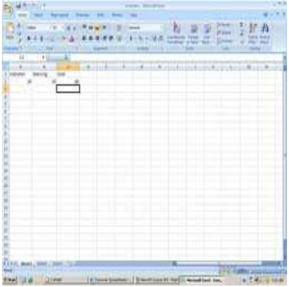




Knowledge models:
$1/1\ 000\ 000 \rightarrow 1/1\ 000 \rightarrow 1/1$
maps

Office of Oceanic and Atmospheric Research (OAR) climate model

Action/decision models:
economic models are fables designed to provide insight

William Nordhaus economic-climate model

This talk is *not* about crafting dynamical models

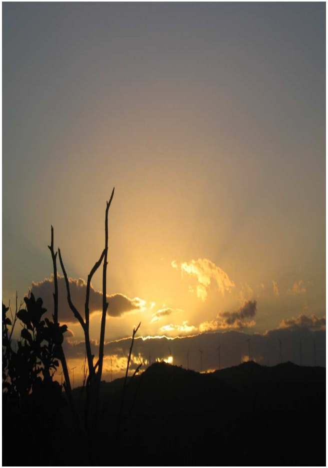
- Elaborating a dynamical model is a delicate venture
  - Peter Yodzis, *Predator-Prey Theory and Management of Multispecies Fisheries*, Ecological Applications 4:51–58, 1994

  In population modelling the functional forms of models are at least as important as are parameter values in expressing the underlying biology and in determining the outcome. (...) For instance, May et al. (1979) assumed, without comment, a particular form of predator-prey interaction; and this particular form was carried over, again without comment, by Flaaten. It turns out that this "invisible" but powerful assumption is responsible in large part for the conclusion reached by Flaaten (1988). (...) Flaaten's work is controversial because of his conclusion that "sea mammals should be heavily depleted to increase the surplus production of fish resources for man" (Flaaten 1988:114).

- Our starting point will be a mathematical dynamical model that captures how sequences of decisions affect a "piece of reality"
- Then, we will use such a model to frame a decision problem

Climate change mitigation

Let us scout a very stylized model of the climate-economy system [De Lara and Doyen, 2008]



We lay out a dynamical model with

- two state variables

  environmental: atmospheric CO_2 concentration level $M(t)$

  economic: gross world product GWP $Q(t)$

- one decision variable, the emission abatement rate $a(t)$

A carbon cycle model "à la Nordhaus" is an example of *decision model*

- Time index t in years
- Economic production $Q(t)$ (GWP)

$$Q(t+1) = \overbrace{(1+g)}^{\text{economic growth}} Q(t)$$

- CO_2 concentration $M(t)$

$$M(t+1) = M(t) \underbrace{-\delta(M(t) - M_{-\infty})}_{\text{natural sinks}} + \alpha \overbrace{\mathtt{Emiss}\big(Q(t)\big)}^{\text{emissions}} \underbrace{\big(1 - a(t)\big)}_{\text{abatement}}$$

- Decision $a(t) \in [0,1]$ is the abatement rate of CO_2 emissions

Data

- $M(t)$ CO_2 atmospheric concentration, measured in ppm, parts per million
  (379 ppm in 2005)
- $M_{-\infty}$ pre-industrial atmospheric concentration
  (about 280 ppm)
- $\texttt{Emiss}(Q(t))$ "business as usual" CO_2 emissions
  (about 7.2 GtC per year between 2000 and 2005)
- $0 \leq a(t) \leq 1$ abatement rate reduction of CO_2 emissions
- α conversion factor from emissions to concentration
  ($\alpha \approx 0.471$ ppm.GtC^{-1} sums up highly complex physical mechanisms)
- δ natural rate of removal of atmospheric CO_2 to unspecified sinks
  ($\delta \approx 0.01$ year^{-1})

A concentration target is pursued to avoid danger

United Nations Framework Convention on Climate Change

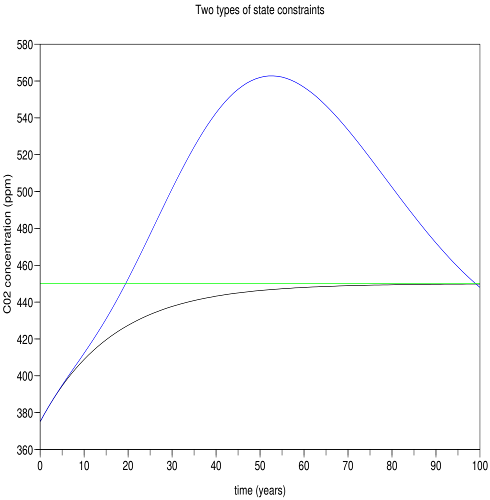
"to achieve, (...), stabilization of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system"

Limitation of concentrations of CO_2

- below a tolerable threshold $M^\sharp$ (say 350 ppm, 450 ppm)
- at a specified date $T > 0$ (say year 2050 or 2100)

$$\underbrace{M(T)}_{\text{concentration at horizon}} \leq \underbrace{M^\sharp}_{\text{threshold}}$$

Constraints capture different requirements



- The concentration has to remain below a tolerable level at the horizon T:

$$M(T) \leq M^{\sharp}$$

- More demanding: from the initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp}$$

$$t = t_0, \ldots, T$$

Constraints may be environmental, physical, economic

- The concentration has to remain below a tolerable level from initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp} , \quad t = t_0, \ldots, T$$

- Abatements are expressed as fractions

$$0 \leq a(t) \leq 1 , \quad t = t_0, \ldots, T-1$$

- As with "cap and trade", setting a ceiling on $\mathrm{CO_2}$ price amounts to cap abatement costs

$$\underbrace{C\big(a(t), Q(t)\big)}_{\text{costs}} \leq c^{\sharp} \, (100 \text{ euros } / \text{ tonne } \mathrm{CO_2}) , \quad t = t_0, \ldots, T-1$$

Mixing dynamics, optimization and constraints yields a cost-effectiveness problem

- Minimize abatement costs

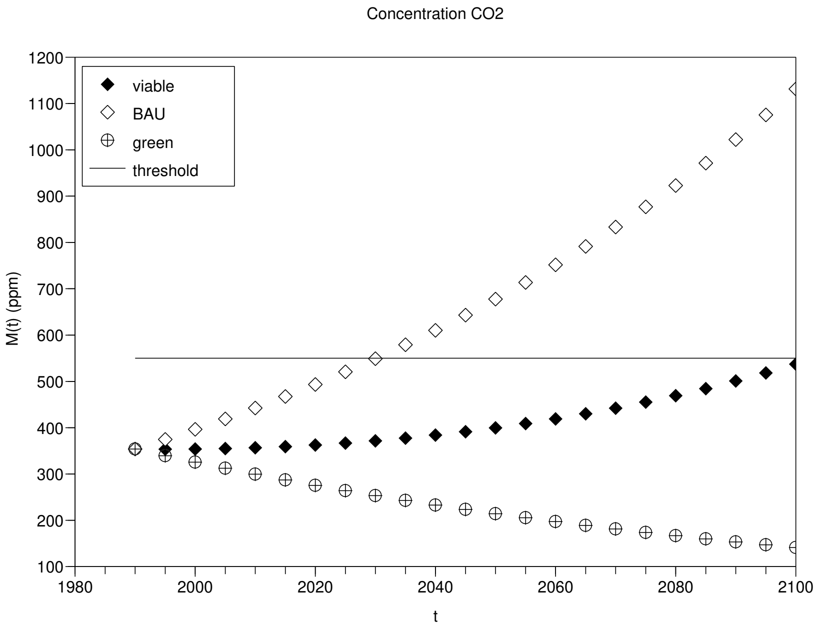
$$\min_{a(t_0),\dots,a(T-1)} \sum_{t=t_0}^{T-1} \left(\frac{1}{1+r_e}\right)^{t-t_0} \underbrace{C\big(a(t), Q(t)\big)}_{\text{abatement costs}}$$

- under the GWP-CO_2 dynamics

$$\begin{cases} M(t+1) &= M(t) - \delta(M(t) - M_{-\infty}) + \alpha \mathtt{Emiss}\big(Q(t)\big)(1-a(t)) \\ Q(t+1) &= (1+g)Q(t) \end{cases}$$

- and under target constraint

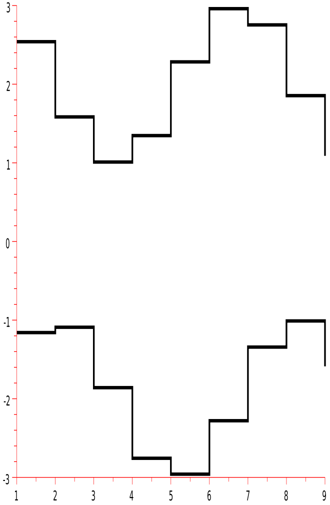
$$\underbrace{M(T) \leq M^\sharp}_{\text{CO2 concentration}}$$



Concentration CO2
viable
BAU
green
threshold
M(t) (ppm)
1200
1100
1000
900
800
700
600
500
400
300
200
100
1980
2000
2020
2040
2060
2080
2100
t

Outline of the presentation

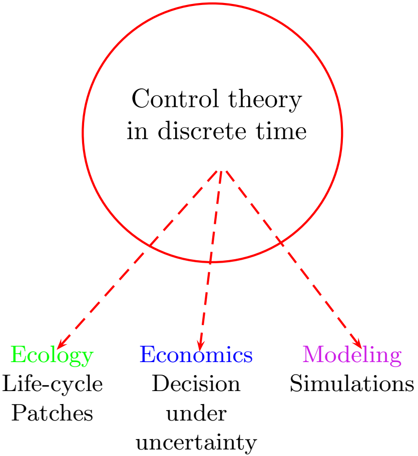
What is resilience?



Resilience is the capacity of a system
to continually change and adapt
yet remain within critical thresholds

Stockholm Resilience Centre

We showcase control theory in discrete time as a proper vehicle for problem formulation [De Lara and Doyen, 2008]

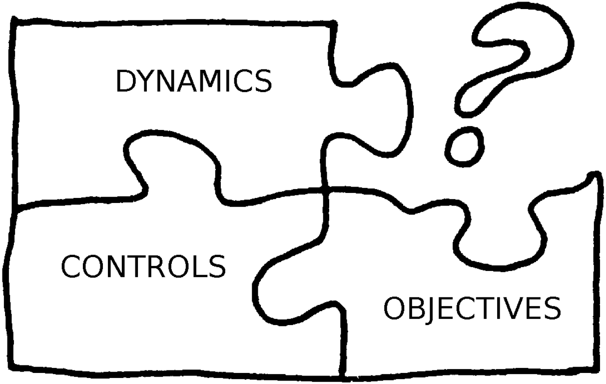


Discrete time nonlinear state-control system

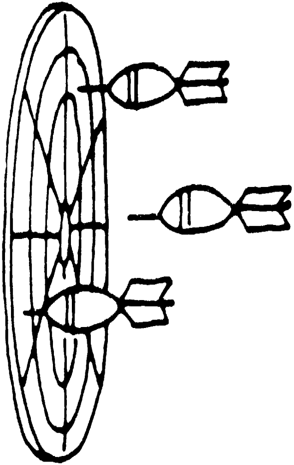
$$x_{t+1} = f_t\big(x_t, u_t\big) , \quad t \in \mathbb{T} = \{t_0, t_0 + 1, \ldots, T - 1\}$$

- the time t (stage) $\in \overline{\mathbb{T}} = \{t_0, t_0 + 1, \ldots, T - 1, T\} \subset \mathbb{N}$ is discrete with initial time t_0 and horizon T ($T < +\infty$ or $T = +\infty$) (*the time period* $[t, t + 1[$ *may be a year, a month, etc.*)
- the state variable x_t belongs to the *state space* $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$ (*stocks, biomasses, abundances, capital*)
- the control variable u_t is an element of the *control space* $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ (*inflows, outflows, catches, harvesting effort, investment*)
- the dynamics f_t maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$ (*storage, age-class model, population dynamics, economic model*)

Viability is relevant to address the compatibility puzzle



We mathematically express the objectives pursued as control and state constraints



- For a state-control system,
  we cloth objectives as constraints
- and we distinguish

  control constraints (rather easy)
  state constraints (rather difficult)
- Viability theory deals with
  state constraints

Constraints may be explicit on the control variable

and are rather easily handled by reducing the decision set

Examples of control constraints

- Irreversibility constraints, physical bounds
  $0 \leq a_t \leq 1$, $0 \leq h_t \leq B_t$
- Tolerable costs $c\big(a_t, Q_t\big) \leq c^\sharp$

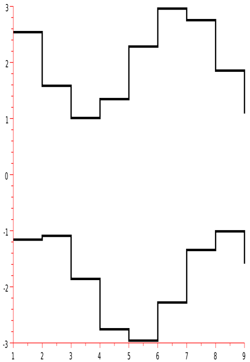
Control constraints / admissible decisions

$$\underbrace{u_t}_{\text{control}} \in \underbrace{\mathbb{B}_t\big(x_t\big)}_{\text{admissible set}} \quad , \quad t = t_0, \ldots, T-1$$

Easy because control variables u_t are precisely those variables
whose values the decision-maker can fix at any time within given bounds

Meeting constraints bearing on the state variable is delicate

due to the dynamics pipeline between controls and state



State constraints / admissible states

$$\underbrace{x_t}_{\text{state}} \in \underbrace{\mathbb{A}_t}_{\text{admissible set}} , \quad t = t_0, \ldots, T$$

Examples ("tipping points")

- CO_2 concentration $M_t \leq M^\sharp$
- biomass $B^\flat \leq B_t \leq B^\sharp$

State constraints are mathematically difficult because of "inertia"

$$x_t = \underbrace{\text{function}}_{\text{iterated dynamics}} \big(\underbrace{u_{t-1}, \ldots, u_{t_0}, x_{t_0}}_{\text{past controls}} \big)$$

Target and asymptotic state constraints are special cases

- Final state achieves some target

$$\underbrace{x_T}_{\text{final state}} \in \underbrace{\mathbb{A}_T}_{\text{target set}}$$

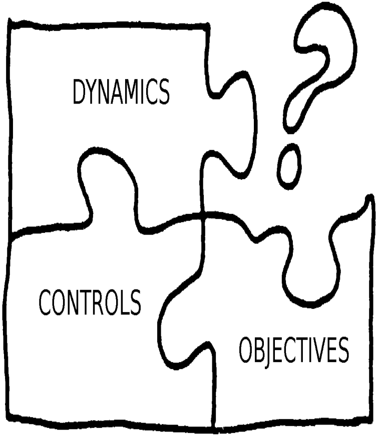
Example: CO_2 concentration

- State converges toward a target

$$\underbrace{\lim_{t \to +\infty} x_t}_{\text{asymptotic state}} \in \underbrace{\mathbb{A}_\infty}_{\text{target set}}$$

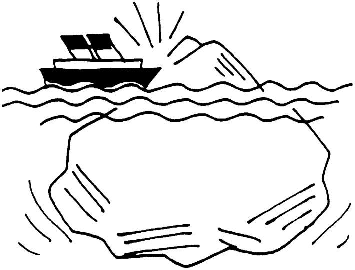
Example: in mathematical epidemiology, convergence towards an endemic state (hence the ubiquitous $\mathcal{R}_0$)

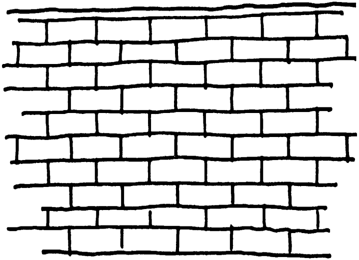
Can we solve the compatibility puzzle between dynamics and objectives by means of suitable controls?



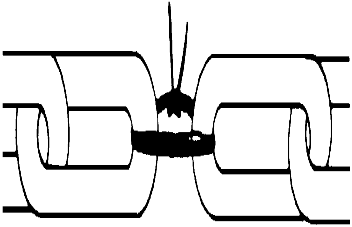
- Given a dynamics that mathematically embodies the causal impact of controls on the state
- Imposing objectives bearing on output variables (states, controls)
- Is it possible to find a control path that achieves the objectives for all times?

Crisis occurs when constraints are trespassed at least once





- An initial state is not viable if, whatever the sequence of controls, a crisis occurs
- There exists a time when one of the state or control constraints is violated



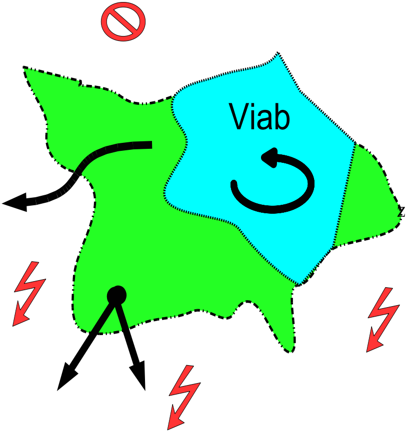
The compatibility puzzle can be solved when the initial viability kernel $\mathbb{V}\mathrm{iab}_{t_0}$ is not empty [Aubin, 1991]

Viable initial states form the viability kernel

$$\mathbb{V}\mathrm{iab}_t = \left\{ \begin{array}{c} \text{initial} \\ \text{states} \\ x \in \mathbb{X} \end{array} \middle| \begin{array}{l} \text{there exist a control path } u(\cdot) = \\ \big(u_t, u_{t+1}, \ldots, u_{T-1}\big) \\ \text{and a state path } x(\cdot) = \\ \big(x_t, x_{t+1}, \ldots, x_T\big) \\ \text{starting from } x_t = x \text{ at time } t \\ \text{satisfying for any time } s \in \{t, \ldots, T-1\} \\ x_{s+1} = f_s(x_s, u_s) \hfill \textit{dynamics} \\ u_s \in \mathbb{B}_s(x_s) \hfill \textit{control constraints} \\ x_s \in \mathbb{A}_s \hfill \textit{state constraints} \\ x_T \in \mathbb{A}_T \hfill \textit{target constraints} \end{array} \right\}$$

J.-P. Aubin. *Viability Theory*. Birkhäuser, Boston, 1991.

The viability kernel is included in the state constraint set



- The largest set is the state constraint set $\mathbb{A}$
- It includes the smaller blue viability kernel $\mathbb{V}\mathrm{iab}_{t_0}$
- The green set measures the incompatibility between dynamics and constraints: good start, but inevitable crisis!

The viability program aims at turning a priori constraints, with state constraints, into a posteriori constraints, without state constraints

- A priori constraints, with state constraints

$$
\begin{cases}
x_{t_0} \in \mathbb{X} & \\
x_{t+1} = f_t(x_t, u_t) & \\
u_t \in \mathbb{B}_t(x_t) & \text{control constraints} \\
x_t \in \mathbb{A}_t & \text{state constraints}
\end{cases}
$$

- are turned into a posteriori constraints, without state constraints except for the initial state

$$
\begin{cases}
x_{t_0} \in \mathbb{V}\mathrm{iab}_{t_0} & \text{initial state constraint} \\
x_{t+1} = f_t\big(x_t, u_t\big) & \\
u_t \in \mathbb{B}_t^{\mathsf{viab}}\big(x_t\big) & \text{control constraints}
\end{cases}
$$

Viable epidemics control

"Canal Endémico" stands as the reference to control dengue

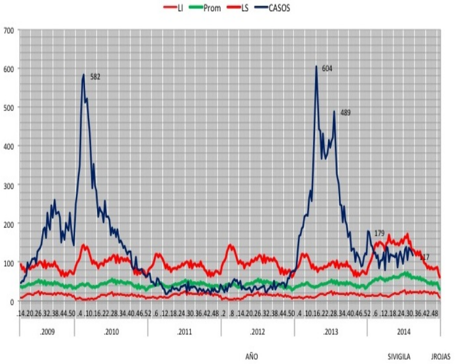


Figure: Cases of dengue between 2009 and 2014. Source: Secretaría Municipal de Salud de Cali.

Program "Dengue Control" of SMS

Control mosquito breeding sites

Capping the human infected population with the Ross-Macdonald model [De Lara and Sepulveda, 2016]

- The dynamics of the system is given by

$$
\begin{array}{lcl}
\text{infected mosquito proportion} & \dfrac{dm}{dt} = & A_m h(t)(1-m(t)) - u(t)m(t) \\[2ex]
\text{infected human proportion} & \dfrac{dh}{dt} = & A_h m(t)(1-h(t)) - \gamma h(t)
\end{array}
$$

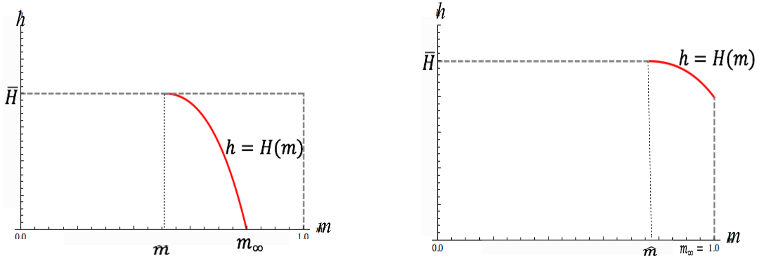
- Determine, if it exists, a piecewise continuous function (fumigation policy rates) $u(\cdot)$,

$$u(\cdot) : t \mapsto u(t) \, , \ \ \underline{u} \leq u(t) \leq \overline{u} \, , \ \ \forall t \geq 0$$

such that the following so-called viability constraint is satisfied

$$h(t) \leq \overline{H} \, , \ \ \forall t \geq 0$$

Capping the human infected population with the Ross-Macdonald model: viability kernels [De Lara and Sepulveda, 2016]



To deal with uncertainties, we sample the controlled Ross–Macdonald model [Sepulveda Salcedo and De Lara, 2019]

$$\big(M_{t+1}, H_{t+1}\big) = f\big(M_t, H_t, u_t, \underbrace{A_t^M, A_t^H}_{\text{uncertainties}}\big)$$

- Basic variables and parameters are
  - time $t = t_0, t_0 + 1 \ldots, T-1, T$, measured in days
  - M_t, the proportion of infected mosquitos (Aedes Aegypti adultos) at the beginning of the day $[t, t+1[$
  - H_t, the proportion of infected humans at the beginning of the day $[t, t+1[$
  - u_t, the mosquito mortality rate (application of chemical control) applied during all day $[t, t+1[$
- The objective is to maintain infected humans at a low level

$$H_t \leq \overline{H} \ , \ \ \forall t = t_0, \ldots, T$$

with limited resources $\underline{u} \leq u_t \leq \overline{u}$, $\forall t = t_0, \ldots, T-1$

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Discrete time nonlinear state-control system

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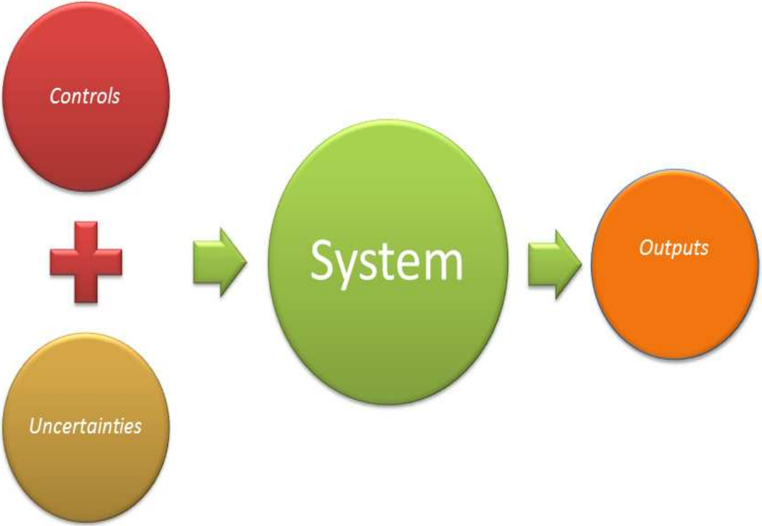
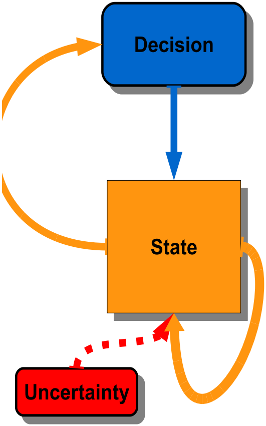
- the state variable x_t belongs to the *state space* $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$ (*stocks, biomasses, abundances, capital*)
- the control variable u_t is an element of the *control space* $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ (*inflows, outflows, catches, harvesting effort, investment*)
- the uncertainty $w_t \in \mathbb{W} = \mathbb{R}^{n_{\mathbb{W}}}$ (*recruitment or mortality uncertainties, climate fluctuations*)
- the dynamics f_t maps $\mathbb{X} \times \mathbb{U} \times \mathbb{W}$ into $\mathbb{X}$ (*storage, age-class model, population dynamics, economic model*)

By contrast with control variables,
uncertainty variables are exogenous input variables



"Policies" are closed-loop controls



- Deterministic control theory appeals to open-loop control, ⌚ that is, a time-dependent sequence (planning, scheduling)

$$u : \underbrace{t \in \mathbb{T}}_{\text{time}} \mapsto \underbrace{u_t \in \mathbb{U}}_{\text{control}}$$

- Another notion of solution is a decision rule, ⌚ × 👁 a policy, that is, a mapping

$$\lambda : \underbrace{(t, x) \in \mathbb{T} \times \mathbb{X}}_{\text{(time, state)}} \mapsto \underbrace{u = \lambda_t(x) \in \mathbb{U}}_{\text{control}}$$

which "closes the loop" between time t–state x and control u (and is especially relevant in presence of uncertainties)

Outline of the presentation

The $(+, \times)$ algebra of probability theory

Probability space

- The set Ω is equipped with a σ-field $\mathcal{F}$ ($(\Omega, \mathcal{F})$ measurable space), and the elements of $\mathcal{F} \subset 2^{\Omega}$ are called events
- One speaks of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ when the measurable space $(\Omega, \mathcal{F})$ is equipped with a probability $\mathbb{P}$ (supposed, when needed and for the sake of simplicity, to have a density p w.r.t. a reference measure, thus covering the finite case)
- The probability $\mathbb{P} : \mathcal{F} \to [0, 1]$ has the properties
  - normalization
  $$\mathbb{P}(\emptyset) = 0 \ , \ \ \mathbb{P}(\Omega) = 1$$
  - additivity
  $$\mathbb{P}(\bigcup_{n \in \mathcal{N}} A_n) = \sum_{n \in \mathcal{N}} \mathbb{P}(A_n)$$
  for any countable set $\mathcal{N}$, $A_n \in \mathcal{F}$ for all $n \in \mathcal{N}$, such that $m \neq n \implies A_m \cap A_m = \emptyset$

Expected value

- A random variable is a measurable mapping $X : (\Omega, \mathcal{F}) \to (\mathbb{X}, \mathcal{X})$ (between measurable spaces)
- The expected value of a nonnegative random variable $X : \Omega \to \mathbb{R}_+ \cup \{+\infty\}$ is

$$\mathbb{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}\mathbb{P}(\omega) \quad \left(\int_\Omega X(\omega)p(\omega)\,\mathrm{d}\omega\right) \quad \left(\sum_{\omega\in\Omega} \mathbb{P}\{\omega\}X(\omega)\right)$$

- The notation $\mathbb{E}$ (or $\mathbb{E}_{\mathbb{P}}$ or $\mathbb{E}^{\mathbb{P}}$) refers to the mathematical expectation (operator) over Ω under probability $\mathbb{P}$, extended to integrable real-valued random variables
- The expectation operator $\mathbb{E}$ enjoys linearity in the $(+, \times)$ algebra

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

- The random variables X, Y are independent under $\mathbb{P}$ when their joint distribution $\mathbb{P}_{(X,Y)}$ can be decomposed as a product

$$\mathbb{P}_{(X,Y)} = \mathbb{P}_X \otimes \mathbb{P}_Y$$

The $(\max, +)$ algebra of decision/robust/plausibility theory

Decision space, cost measure, plausibility are the robust counterparts of probability space

- The set Ω is equipped with a σ-field $\mathcal{F}$ ($(\Omega, \mathcal{F})$ measurable space)
- One speaks of a decision space $(\Omega, \mathcal{F}, \mathbb{K})$ when the measurable space $(\Omega, \mathcal{F})$ is equipped with a cost measure $\mathbb{K}$ (supposed, when needed, to have a density κ, thus covering the finite case)
- The cost measure (plausibility) $\mathbb{K} : \mathcal{F} \to [-\infty, 0]$ has the properties
  - normalization
    $$\mathbb{K}(\emptyset) = -\infty \, , \ \ \mathbb{K}(\Omega) = 0$$
  - $(\max, +)$ "additivity"
    $$\mathbb{K}(\bigcup_{n \in \mathcal{N}} A_n) = \sup_{n \in \mathcal{N}} \mathbb{K}(A_n)$$
    for any countable set $\mathcal{N}$, $A_n \in \mathcal{F}$ for all $n \in \mathcal{N}$, such that $m \neq n \implies A_m \cap A_m = \emptyset$

Cost density, plausibility function

- The function $\kappa : \Omega \to [-\infty, 0]$ is a cost density of the cost measure $\mathbb{K}$ if

$$\mathbb{K}(A) = \sup_{\omega \in A} \kappa(\omega) \ , \ \ \forall A \in \mathcal{F}$$

- A function $\kappa : \Omega \to [-\infty, 0]$,
such that $\sup_{\omega \in \Omega} \kappa(\omega) = 0$,
is a cost density, also called plausibility function

The fear operator [Bernhard, 1995]

The Moreau lower addition extends the usual addition with

$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = -\infty$$

- A decision variable is a mapping $(\Omega, \mathcal{F}) \to (\mathbb{T}, \mathcal{T})$ (with codomain a topological space)
- The feared value of a function $\psi : \Omega \to [-\infty, +\infty]$ (real-valued decision variable) is defined by

$$\mathbb{F}(\psi) = \sup_{\omega \in \Omega} \left[\psi(\omega) \dot{+} \kappa(\omega) \right]$$

- The fear operator $\mathbb{F}$ enjoys linearity in the $(\max, +)$ algebra

$$\mathbb{F}\big(\max\{\psi, \phi\}\big) = \max\{\mathbb{F}(\psi), \mathbb{F}(\phi)\}$$

- Independence

$$\mathbb{K}_{(\psi,\phi)} = \mathbb{K}_\psi + \mathbb{K}_\phi$$

Two applications of the parallelism between $(+, \times)$ and $(\max, +)$ algebras

Magic formulas in optimization

Nested optimization / Tower formula

$$\inf_{(a,b)\in\mathbb{A}\times\mathbb{B}} h(a,b) = \inf_{a\in\mathbb{A}} \Big(\inf_{b\in\mathbb{B}} h(a,b)\Big)$$

$$\mathbb{E}[h(\mathsf{A},\mathsf{B})] = \mathbb{E}\big[\mathbb{E}[h(\mathsf{A},\mathsf{B}) \mid \mathsf{A}]\big]$$

Decomposition, parallel optimization / Independence

$$\inf_{(a,b)\in\mathbb{A}\times\mathbb{B}} \big(f(a) + g(b)\big) = \inf_{a\in\mathbb{A}} f(a) + \inf_{b\in\mathbb{B}} g(b)$$

$$\mathsf{A},\mathsf{B} \text{ independent} \implies \mathbb{E}[f(\mathsf{A}) \times g(\mathsf{B})] = \mathbb{E}[f(\mathsf{A})] \times \mathbb{E}[g(\mathsf{B})]$$

More or less implausible events

- For any subset $\Omega'' \subset \Omega$, we have that

$$\mathbb{K}(\emptyset) = -\infty \leq \mathbb{K}(\Omega'') \leq \mathbb{K}(\Omega) = 0$$

- The higher (closest to zero from below), the more plausible, whereas totally implausible outcomes in Ω'' are such that $\mathbb{K}(\Omega'') = -\infty$
- With any subset $\overline{\Omega} \subset \Omega$, we associate the characteristic function

$$\delta_{\overline{\Omega}}(\omega) = \begin{cases} 0 & \text{if } \omega \in \overline{\Omega} \\ +\infty & \text{if } \omega \notin \overline{\Omega} \end{cases}$$

- The cost measure $\mathbb{K}$ associated with the uniform density $-\delta_{\overline{\Omega}}$ satisfies, for any subset $\Omega' \subset \Omega$,

$$\mathbb{K}(\Omega \setminus \Omega') = \sup_{\omega \in \Omega \setminus \Omega'} \left(-\delta_{\overline{\Omega}}(\omega)\right) = \begin{cases} -\infty & \text{if } \overline{\Omega} \subset \Omega' \\ 0 & \text{if } (\Omega \setminus \Omega') \cap \overline{\Omega} \neq \emptyset \end{cases}$$

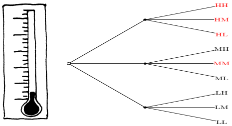
Outline of the presentation

We call scenario a temporal sequence of uncertainties

Scenarios are special cases of “states of Nature”

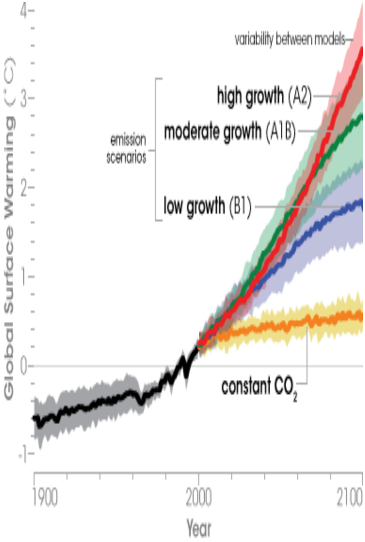
A scenario (pathway, chronicle) is a sequence of uncertainties

$$w(\cdot) = \big(w_{t_0}, \ldots, w_{T-1}\big) \in \mathbb{S} = \mathbb{W}^{T-t_0}$$



El tiempo se bifurca perpetuamente hacia innumerables futuros
(Jorge Luis Borges, *El jardín de senderos que se bifurcan*)

Beware! Scenario holds a different meaning in other scientific communities



- In practice, what modelers call a "scenario" is a mixture of
  - a sequence of uncertain variables (also called a pathway, a chronicle)
  - a policy
  - and even a static or dynamical model
- In what follows

  scenario = pathway = chronicle

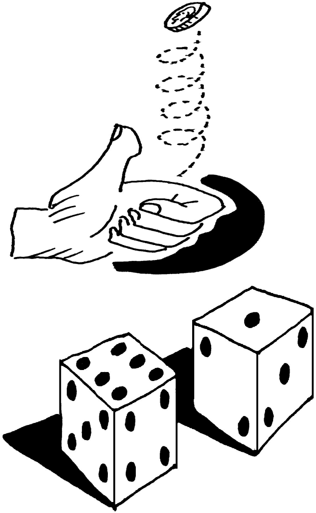
Choosing a set of scenarios is excluding "things we don't know we don't know"

> *Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.*

Donald Rumsfeld, former United States Secretary of Defense.
From Department of Defense news briefing, February 12, 2002

Scenarios
stochastic *vs* robust

In the stochastic approach, the set of scenarios is equipped with a known probability



A priori information on the scenarios may be probabilistic

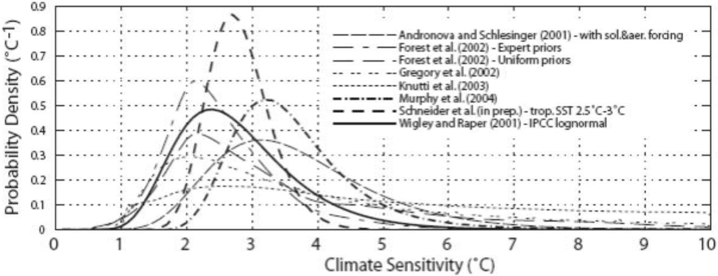
- A probability distribution $\mathbb{P}$ on $\mathbb{S}$
- In practice, one often assumes that the components $(w_{t_0}, \ldots, w_{T-1})$ form
  - an independent and identically distributed sequence
  - a Markov chain, a time series, etc.

Water inflows in a dam

Water inflows in a dam may be modelled as time series (ARMA, etc.)

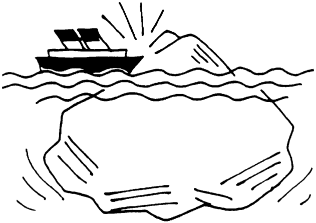
Equipping the set $\mathbb{S}$ of scenarios with a probability $\mathbb{P}$ is a delicate issue!

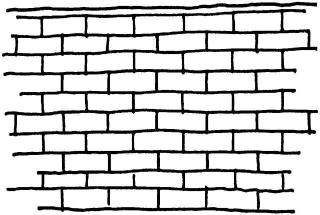
- The probabilistic distribution of the climate sensitivity parameter in climate models differs according to authors

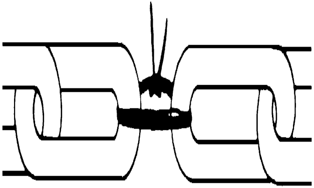


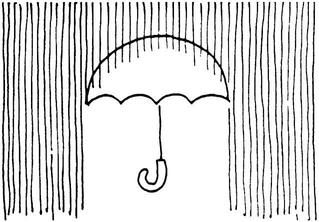
- In the multi-prior approach, the a priori information consists of different probabilities (*beliefs*, *priors*), belonging to a set $\mathcal{P}$ of admissible probabilities on $\mathbb{S}$

In the set-membership approach,
only a subset of the set of scenarios is known







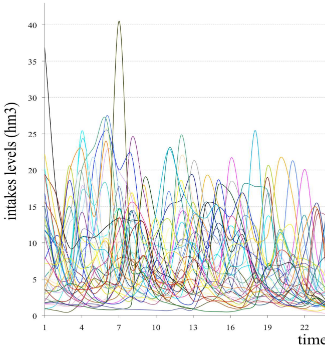


A priori information on the scenarios may be set membership

The general case

- Selected scenarios may belong to any subset $\overline{\mathbb{S}}$

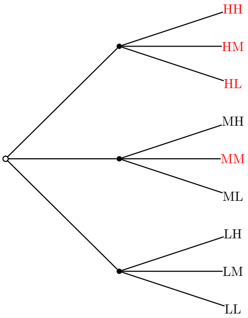
$$w(\cdot) \in \overline{\mathbb{S}} \subset \mathbb{S}$$



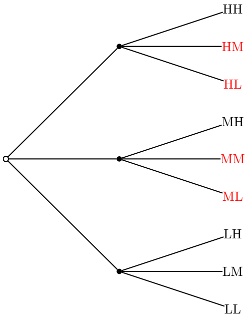
Historical water inflows scenarios in a dam

We can represent offline information by the observed historical water inflows scenarios

Specific subsets correspond to time independence



NO time independence because
the range of values of w_{t+1} depends
on the value of w_t:
$w_t = H \Rightarrow w_{t+1} \in \{H, M, L\}$
$w_t = M \Rightarrow w_{t+1} \in \{M\}$



Time independence because
$\overline{\mathbb{S}} = \{H, M\} \times \{M, L\} \subset \mathbb{S}$
is a product set

A priori information on the scenarios may be softer than set membership thanks to plausibility functions

Plausibility function $\kappa : \mathbb{S} \to \mathbb{R}_- \cup \{-\infty\}$
such that (normalization)

$$\sup_{w(\cdot)\in\mathbb{S}} \kappa\big(w(\cdot)\big) = 0$$

can "soften" the above set membership approach

- the higher $\kappa\big(w(\cdot)\big)$, the more plausible the scenario $w(\cdot)$
- totally implausible scenarios are those for which $\kappa\big(w(\cdot)\big) = -\infty$

Historical water inflows scenarios in a dam

Attribute the value $\kappa\big(w(\cdot)\big) = -\infty$ for all the scenarios $w(\cdot)$ which do not belong to the observed historical water inflows scenarios

Summary

- A priori information is carried by the scenarios set, and may be
  - probabilistic (probability and expectation operator)
  - set membership (plausibility and fear operator)
- This will be useful to mathematically express
  objectives and constraints
  in a decision problem under uncertainty

Outline of the presentation

The viability approach

Handling uncertainty in control theory

The stochastic/robust viability approaches

Measures of resilience and extensions

"Self-promotion, nobody will do it for you" ;-)

Outline of the presentation

A scenario is said to be viable for a given policy if the state and control trajectories satisfy the constraints

Viable scenario under given policy

A scenario $w(\cdot) \in \mathbb{S}$ is said to be viable under policy $\lambda : \mathbb{T} \times \mathbb{X} \to \mathbb{U}$ if the trajectories $x(\cdot)$ and $u(\cdot)$ generated by the dynamics

$$x_{t+1} = f_t\big(x_t, u_t, w_{t+1}\big) , \quad t = t_0, \dots, T-1$$

with the policy

$$u_t = \lambda_t\big(x_t\big)$$

satisfy the state and control constraints

$$\underbrace{u_t \in \mathbb{B}_t\big(x_t\big)}_{\text{control constraints}} \quad \text{and} \quad \underbrace{x_t \in \mathbb{A}_t}_{\text{state constraints}} \quad , \quad \forall t = t_0, \dots, T$$

The set of viable scenarios is denoted by $\mathbb{S}^{\lambda}_{t_0,x_0}$

We look after policies that make the corresponding set of viable scenarios "large"

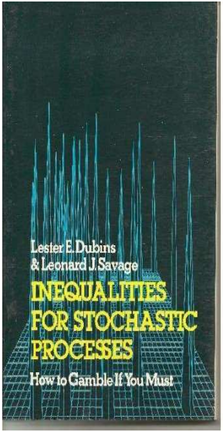
Set of viable scenarios

$$\mathbb{S}^{\lambda}_{t_0,x_0} = \{ w(\cdot) \in \mathbb{S} \mid \quad \text{the state constraints}$$
$$x_t \in \mathbb{A}_t$$
$$\text{and the control constraints}$$
$$u_t = \lambda_t\big(x_t\big) \in \mathbb{B}_t\big(x_t\big)$$
$$\text{are satisfied for all times } t = t_0, \ldots, T \}$$

- The larger set $\mathbb{S}^{\lambda}_{t_0,x_0}$ of viable scenarios, the better, because the policy λ is able to maintain the system within constraints for a large "number" of scenarios
- But "large" in what sense? Probabilistic (stochastic)? Robust?

Outline of the presentation

Maximizing the probability of success may be an objective



How to gamble if you must, L.E. Dubbins and L.J. Savage, 1965

Imagine yourself at a casino with \$1,000. For some reason, you desperately need \$10,000 by morning; anything less is worth nothing for your purpose.

The only thing possible is to gamble away your last cent, if need be, in an attempt to reach the target sum of \$10,000.

- The question is how to play, not whether. What ought you do? How should you play?
  - Diversify, by playing 1 \$ at a time?
  - Play boldly and concentrate, by playing 1,000 \$ only one time?
- What is your decision criterion?

We extend viability kernels to
stochastic viability kernels

Stochastic viability kernels

In stochastic viability, state constraints are to be met along time with a given confidence level $\beta \in [0,1]$

$$\mathbb{P}\Big(w(\cdot) \in \mathbb{S} \mid x_t \in \mathbb{A}_t \ , \ \ u_t = \lambda_t\big(x_t\big) \in \mathbb{B}_t\big(x_t\big) \text{ for } t = t_0, \ldots, T \Big) \geq \beta$$

or, equivalently,

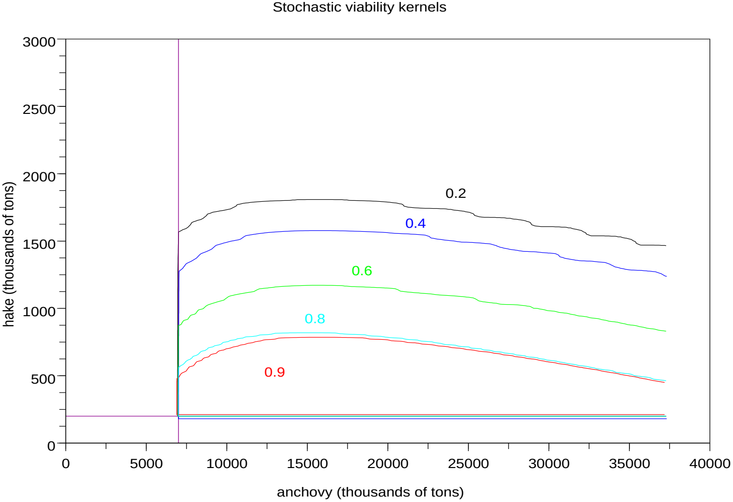
$$\mathbb{P}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0} \Big) \leq 1 - \beta$$

Stochastic viability kernels

The stochastic viability kernel at confidence level $\beta \in [0,1]$ is

$$\mathbb{V}\mathrm{iab}^{\beta}_{t_0} = \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \text{there exists a policy } \lambda \text{ such that} \\ \mathbb{P}\Big(w(\cdot) \in \mathbb{S} \mid x_t \in \mathbb{A}_t \ , \ \ u_t = \lambda_t\big(x_t\big) \in \mathbb{B}_t\big(x_t\big) \\ \ \text{for } t = t_0, \ldots, T \Big) \geq \beta \end{array} \right\}$$

Stochastic viability kernels $\mathbb{V}\mathrm{iab}_{t_0}^{\beta}$ for a hake-anchovy fisheries model [De Lara, Martinet, and Doyen, 2015]



Stochastic viability kernels
can be obtained by
dynamic programming
[Doyen and De Lara, 2010]

The viability probability is the probability of satisfying constraints under a policy

Viability probability

The viability probability associated with
the initial time t_0, the initial state x_0 and the policy λ
is the probability $\mathbb{P}\big(\mathbb{S}^{\lambda}_{t_0,x_0}\big)$ of the set $\mathbb{S}^{\lambda}_{t_0,x_0}$ of viable scenarios

$$\mathbb{P}\big(\mathbb{S}^{\lambda}_{t_0,x_0}\big) = \mathbb{P}\{w(\cdot) \in \mathbb{S} \mid$$

$$\text{the state constraints } \; x_t \in \mathbb{A}_t$$

$$\text{and the control constraints } \; u_t = \lambda_t(x_t) \in \mathbb{B}_t(x_t)$$

$$\text{are satisfied for all times } \; t = t_0, \dots, T\}$$

The maximal viability probability is the upper bound for the probability of satisfying constraints

Maximal viability probability and optimal viable policy

The maximal viability probability is

$$\max_{\lambda} \mathbb{P}\Big(\mathbb{S}^{\lambda}_{t_0,x_0}\Big) = 1 - \min_{\lambda} \mathbb{P}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big)$$

An optimal viable policy λ^* satisfies

$$\mathbb{P}\Big(\mathbb{S}^{\lambda^*}_{t_0,x_0}\Big) \geq \mathbb{P}\Big(\mathbb{S}^{\lambda}_{t_0,x_0}\Big)$$

In a sense, any optimal viable policy makes the set of viable scenarios the "largest" possible

Let us introduce the stochastic viability Bellman function

Suppose that the primitive random variables
$\big(w_{t_0}, w_{t_0+1}, \dots, w_{T-2}, w_{T-1}\big)$
are independent under the probability $\mathbb{P}$

Bellman function / stochastic viability value function

Define the probability-to-go as

$$V_t(x) =$$

$$\max_{\lambda} \mathbb{P}\Big(w(\cdot) \in \mathbb{S} \mid \overbrace{\lambda_s\big(x_s\big) \in \mathbb{B}_s\big(x_s\big)}^{\text{control constraints}} \text{ and } \overbrace{x_s \in \mathbb{A}_s}^{\text{state constraints}} \text{ for } s \geq t \Big)$$

where $x_{s+1} = f_s\big(x_s, \lambda_s\big(x_s\big), w_{s+1}\big)$ and $x_t = x$

- The function $V_t(x)$ is called stochastic viability value function (Bellman function)
- The original problem is $V_{t_0}\big(x_0\big)$

The dynamic programming equation is a backward equation satisfied by the stochastic viability value function

Proposition

If the primitive random variables $\big(w_{t_0}, w_{t_0+1}, \ldots, w_{T-2}, w_{T-1}\big)$ *are independent under the probability* $\mathbb{P}$*,*
the stochastic viability value functions $V_{t_0}, \ldots, V_T$
satisfy the following backward induction

$$
\begin{aligned}
V_T(x) &= 1_{\mathbb{A}_T}(x) \\
V_t(x) &= 1_{\mathbb{A}_t}(x) \max_{u \in \mathbb{B}_t(x)} \mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]
\end{aligned}
$$

for all $x \in \mathbb{X}$*, and where* t *runs from* $T-1$ *down to* t_0

Algorithm for the Bellman functions and the stochastic viable controls

initialization $V_T(x) = 1_{\mathbb{A}_T}(x)$;
for $t = T, T-1, \dots, t_0$ **do**
  forall $x \in \mathbb{X}$ **do**
    forall $u \in \mathbb{B}_t(x)$ **do**
      $\mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]$
    $\max_{u \in \mathbb{B}(t,x)} \mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]$
  $V_t(x) = 1_{\mathbb{A}_t}(x) \max_{u \in \mathbb{B}_t(x)} \mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]$

The stochastic viable dynamic programming equation yields stochastic viable policies

For any time t and state x, let us assume that the set

$$\mathbb{B}_t^{\text{viab}}(x) = \arg\max_{u \in \mathbb{B}_t(x)} \Big(1_{\mathbb{A}_t}(x) \mathbb{E}_{w_{t+1}} \Big[V_{t+1}\Big(f_t(x, u, w_{t+1}) \Big) \Big] \Big)$$

of viable controls is not empty

Proposition

Then, any (measurable) policy λ such that $\lambda_t^*(x) \in \mathbb{B}_t^{viab}(x)$ is an optimal viable policy which achieves the maximal viability probability

$$V_{t_0}(x_0) = \max_{\lambda} \mathbb{P}\Big(\mathbb{S}_{t_0, x_0}^{\lambda} \Big)$$

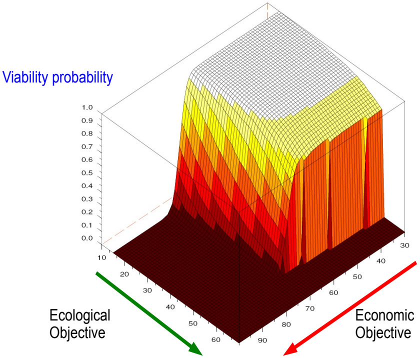
The dynamic programming equation yields the stochastic viability kernels

The stochastic viability kernel at confidence level β turns out to coincide with the section of level β of the stochastic value function:

$$V_{t_0}(x_0) \geq \beta \iff x_0 \in \mathbb{V}\mathrm{iab}_{t_0}^{\beta}$$

Displaying trade-offs between critical thresholds and risk [De Lara and Martinet, 2009]

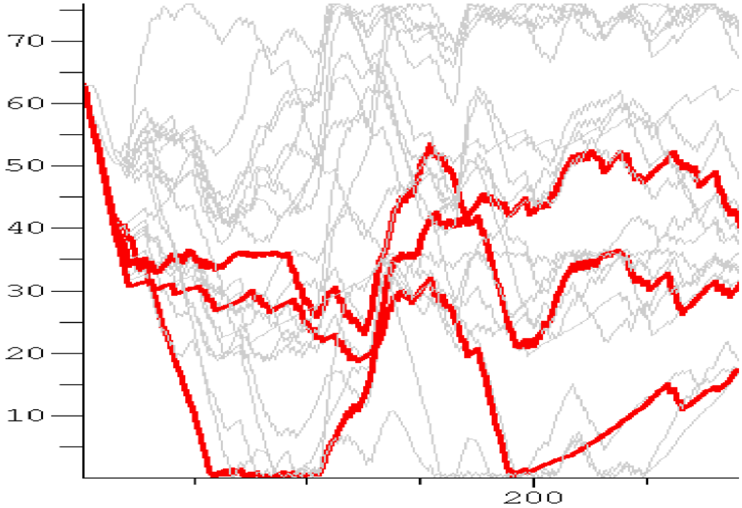
$$\mathbb{P}\left[\underbrace{C_t \geq C^\flat,\ E_t \geq E^\flat}_{\text{indicators} \,\geq\, \text{thresholds}} \;,\ \forall t\right]$$



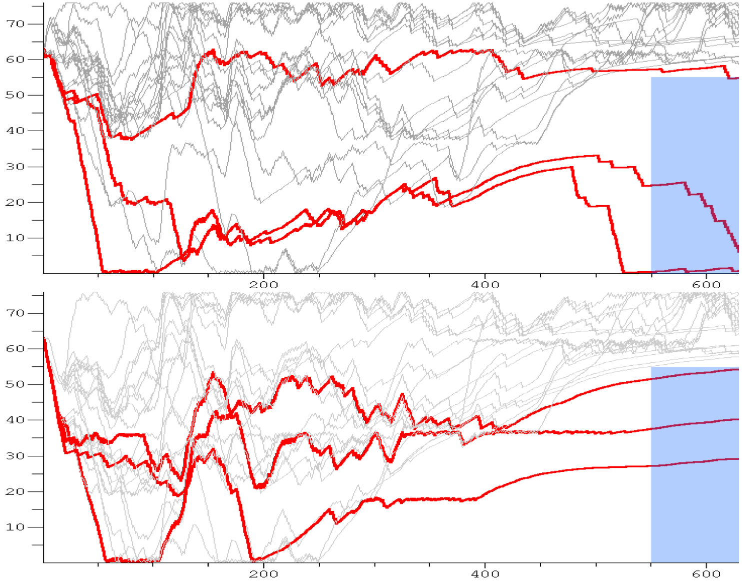
Tourism issues impose constraints upon traditional economic management of a hydro-electric dam [Alais, Carpentier, and De Lara, 2017]

- Maximizing the revenue from turbinated water
- under a tourism constraint of having enough water in July and August

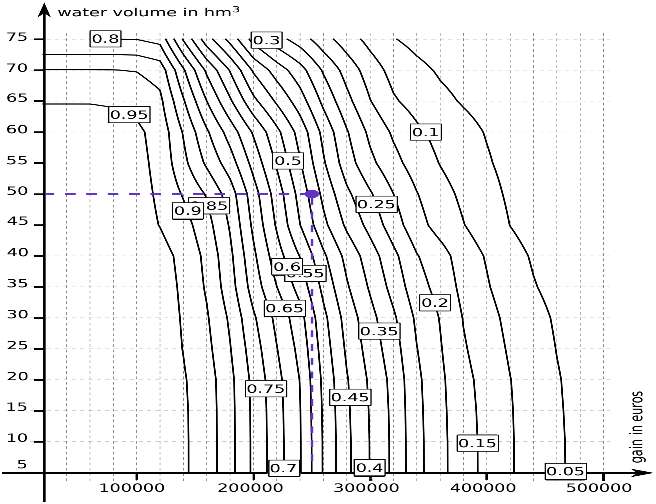
The red stock trajectories fail to meet the tourism constraint in July and August



90% of the stock trajectories meet the tourism constraint in July and August



We plot iso-values for the maximal viability probability as a function of guaranteed thresholds $S^\flat$ and $P^\flat$



Outline of the presentation

The viability approach
- A few words on the purpose of modelling
- (Deterministic) viability in a nutshell

Handling uncertainty in control theory
- Discrete time nonlinear state-control system
- $(+, \times)$ and $(\max, +)$ algebras
- Scenarios/uncertainty chronicles

The stochastic/robust viability approaches
- Viable scenarios
- Stochastic viability in a nutshell
- Robust viability in a nutshell

Measures of resilience and extensions
- How to measure resilience?
- From viable states to viable random paths

"Self-promotion, nobody will do it for you" ;-)

We extend viability kernels to
robust viability kernels

Beware: what is below is under construction

Robust viability kernels

In robust viability, state constraints that are not met along time with a given implausibility level $\eta \in [-\infty, 0]$

$$\mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big) \leq \eta$$

Robust viability kernels

The robust viability kernel at implausibility level $\eta \in [-\infty, 0]$ is

$$\mathbb{V}\mathrm{iab}^{\eta}_{t_0} = \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \text{there exists a policy } \lambda \text{ such that} \\ \mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big) \leq \eta \end{array} \right\}$$

We recover the classic robust framework by using a uniform density

- The classic robust framework $\overline{\mathbb{S}} \subset \mathbb{S}^{\lambda}_{t_0,x_0}$
- corresponds to the cost measure $\mathbb{K}$
  associated with the uniform density $-\delta_{\overline{\mathbb{S}}}$ because

$$\mathbb{K}(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}) = \sup_{w(\cdot)\in\mathbb{S}\setminus\mathbb{S}^{\lambda}_{t_0,x_0}} \big(-\delta_{\overline{\mathbb{S}}}(w(\cdot))\big) = \begin{cases} -\infty & \text{if } \overline{\mathbb{S}} \subset \mathbb{S}^{\lambda}_{t_0,x_0} \\ & (\mathbb{S}^{\lambda}_{t_0,x_0} \text{ totally plausible}) \\ 0 & \text{if } (\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}) \cap \overline{\mathbb{S}} \neq \emptyset \end{cases}$$

- so that

$$\mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big) \leq \eta < 0 \iff \overline{\mathbb{S}} \subset \mathbb{S}^{\lambda}_{t_0,x_0}$$

Robust viability kernels
can be obtained by
dynamic programming

The viability implausibility is the implausibility of satisfying constraints under a policy

Viability implausibility

The viability implausibility associated with
the initial time t_0, the initial state x_0 and the policy λ
is the implausibility $\mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big)$ of the set $\mathbb{S}^{\lambda}_{t_0,x_0}$ of viable scenarios

$$\mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big) = \mathbb{K}\big(\mathbb{S} \setminus \{w(\cdot) \in \mathbb{S} \mid$$

$$\text{the state constraints } x_t \in \mathbb{A}_t$$

$$\text{and the control constraints } u_t = \lambda_t\big(x_t\big) \in \mathbb{B}_t\big(x_t\big)$$

$$\text{are satisfied for all times } t = t_0, \ldots, T\} \big)$$

The minimal viability implausibility is the lower bound for the implausibility of satisfying constraints

Minimal viability implausibility and optimal viable policy

The minimal viability implausibility is

$$\min_{\lambda} \mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big)$$

An optimal viable policy λ^* satisfies

$$\mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda^*}_{t_0,x_0}\Big) \leq \mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big)$$

In a sense, any optimal viable policy makes the set of viable scenarios the "largest" possible

The viability implausibility is (max, +)-additive

$$\mathbb{K}\Big(\mathbb{S} \setminus \mathbb{S}^{\lambda}_{t_0,x_0}\Big) = \mathbb{K}\big(\mathbb{S} \setminus \{w(\cdot) \in \mathbb{S} \mid$$

$$\text{the state constraints } \; x_t \in \mathbb{A}_t$$

$$\text{and the control constraints } \; u_t = \lambda_t\big(x_t\big) \in \mathbb{B}_t\big(x_t\big)$$

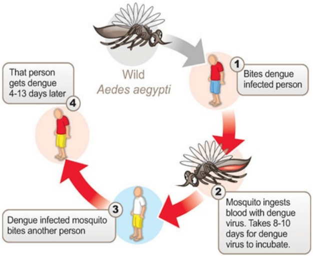
$$\text{are satisfied for all times } t = t_0, \ldots, T\} \;\big)$$

$$= \mathbb{K}\Big(\bigcup_{t=t_0}^{T} \{w(\cdot) \in \mathbb{S} \mid x_t \not\in \mathbb{A}_t \; \text{ or } \; u_t = \lambda_t\big(x_t\big) \not\in \mathbb{B}_t\big(x_t\big)\}\Big)$$

$$= \max_{t=t_0,\ldots,T} \mathbb{K}\Big(\{w(\cdot) \in \mathbb{S} \mid x_t \not\in \mathbb{A}_t \; \text{ or } \; u_t = \lambda_t\big(x_t\big) \not\in \mathbb{B}_t\big(x_t\big)\}\Big)$$

Robust viable epidemics control
[Sepulveda Salcedo and De Lara, 2019]

Sources of uncertainty abound



Uncertainties are captured by $\begin{cases} \text{mosquitoes transmission rate} & A_t^M \\ \text{human transmission rate} & A_t^H \end{cases}$

in the forthcoming model

New variables

- Time
  - Discrete-time $t = 0, 1, \ldots, T$
    with interval $[t, t+1[$ representing one day
- State variables
  - M_t denotes the proportion of infected mosquitoes
    at the beginning of the interval $[t, t+1[$
  - H_t denotes the proportion of infected humans
    at the beginning of the interval $[t, t+1[$
- Control variable
  - U_t denotes the mosquito mortality due to fumigation
    during the interval $[t, t+1[$

Discrete-time dynamic control model with uncertainties

- Let us denote by $f(M, H, U, A^M, A^H)$ the solution, at time $s = 1$, of the deterministic differential system with initial condition $\big(m_0, h_0\big) = (M, H)$ and stationary control U
- We obtain the sampled and controlled Ross–Macdonald model

$$\big(M_{t+1}, H_{t+1}\big) = f\big(M_t, H_t, U_t, A^M_t, A^H_t\big)$$

- The control constraints capture limited fumigation resources

$$\underline{U} \leq U_t \leq \overline{U} \ , \ \ \forall t = 0, \ldots, T-1$$

during a day

Viability problem statement

- We impose that the viability constraint

$$H_t \leq \overline{H} \ , \ \ \forall t = 0, \ldots, T$$

- holds true whatever the scenario (sequence of uncertainties)

$$(A^M(\cdot), A^H(\cdot)) = \Big((A_0^M, A_0^H), \ldots, (A_{T-1}^M, A_{T-1}^H) \Big)$$

belonging to a subset $\overline{\mathbb{S}} \subset (\mathbb{R}^2)^T$

In the robust framework, we need a new definition of solution

- A policy Λ is defined as a sequence of mappings

$$\Lambda = \{\Lambda_t\}_{t=0,\dots,T-1} \,, \quad \text{with} \quad \Lambda_t : [0,1]^2 \to \mathbb{R}$$

  where each Λ_t maps state (M, H) towards control U
- A policy induces a sequence of controls by

$$U_t = \Lambda_t\big(M_t, H_t\big)$$

- A policy Λ is said to be admissible
  if it satisfies the control constraints

$$\Lambda_t : [0,1]^2 \to [\underline{U}, \overline{U}]$$

Robust viability problem statement

The robust viability kernel is the set of initial conditions (M_0, H_0) from which at least one admissible policy Λ gives infected mosquitoes and infected humans trajectories by the dynamics

$$\big(M_{t+1}, H_{t+1}\big) = f\big(M_t, H_t, U_t, A_t^M, A_t^H\big)$$

with input controls

$$U_t = \Lambda_t\big(M_t, H_t\big)$$

so that

$$H_t \leq \overline{H}\ , \ \ \forall t = 0, \ldots, T$$

for all the scenarios

$$\Big(\big(A_0^M, A_0^H\big), \ldots, \big(A_{T-1}^M, A_{T-1}^H\big)\Big) \in \overline{\mathbb{S}} \subset (\mathbb{R}^2)^T$$

We make a tough assumption on the set of scenarios

- A scenario is a time sequence of uncertainty couples

$$\big(A^M(\cdot), A^H(\cdot)\big) = \Big(\big(A^M_0, A^H_0\big), \ldots, \big(A^M_{T-1}, A^H_{T-1}\big)\Big)$$

- We make the strong independence assumption that

$$\big(A^M_t(\cdot), A^H(\cdot)\big) \in \overline{\mathbb{S}} = \mathbb{S}_0 \times \mathbb{S}_1 \times \cdots \times \mathbb{S}_{T-1}$$

- Therefore, from one time t to the next $t+1$, uncertainties can be drastically different since (A^M_t, A^H_t) is not related to (A^M_{t+1}, A^H_{t+1})
- Such an assumption makes it possible to write a dynamic programming equation with (M, H) as state variable
- For the sake of simplicity, we take

$$\mathbb{S}_0 = \mathbb{S}_1 = \cdots = \mathbb{S}_{T-1} = \mathbb{S}$$

Numerical resolution of the dynamic programming equation

initialization $\mathrm{V}_T(M,H) = 1_{[0,1]\times[0,\overline{H}]}(M,H)$;

for $t = T, T-1, \dots, 0$ **do**

  forall $(M,H) \in [0,1]\times[0,\overline{H}]$ **do**

    forall $U \in [\underline{U}, \overline{U}]$ **do**

      forall $(A^M, A^H) \in \mathbb{S}$ **do**

        $\mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$

      $\min_{(A^M,A^H)\in\mathbb{S}} \mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$

    $\max_{U\in[\underline{U},\overline{U}]} \min_{(A^M,A^H)\in\mathbb{S}} \mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$

  $\mathrm{V}_t\big(t,M,H\big) = 1_{[0,1]\times[0,\overline{H}]}(M,H) \times \mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$

Uncertainty sets

We consider three nested sets of uncertainties

$$\mathbb{S}_L \subset \mathbb{S}_M \subset \mathbb{S}_H \subset \mathbb{R}^2_+$$

L) deterministic case

$$\mathbb{S}_L = \left\{ \widehat{A^M} \right\} \times \left\{ \widehat{A^H} \right\}$$

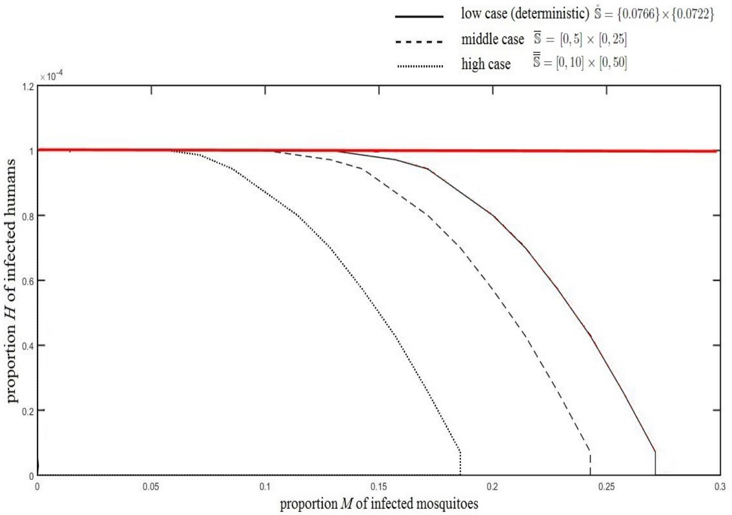
M) medium case

$$\mathbb{S}_M = \left[\underline{A^M}, \overline{A^M} \right] \times \left[\underline{A^H}, \overline{A^H} \right]$$

H) high case

$$\mathbb{S}_H = \left[\underline{\underline{A^M}}, \overline{\overline{A^M}} \right] \times \left[\underline{\underline{A^H}}, \overline{\overline{A^H}} \right]$$

Robust viability kernels shrink when uncertainties expand



Conclusion on robust viability analysis

The numerical results show that the viability kernel without uncertainties is highly sensitive to the variability of parameters such as

- biting rate
- probability of infection to mosquitoes and humans
- proportion of female mosquitoes per person

Maybe we should focus the effort on reducing
these three sources of uncertainty

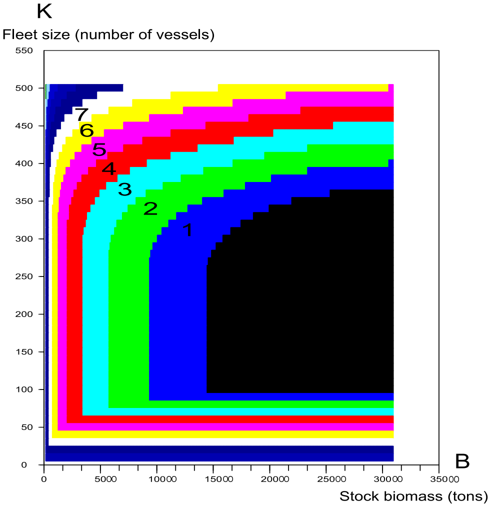
Outline of the presentation

Outline of the presentation

Exposure, vulnerability, resilience?

- Acceptable set/viability constraints:
  - possible values for output variables + critical thresholds
- Adaptive capacity: set of viable policies?
  - = policies depending on available observations and enabling the system to remain within the acceptable set for a certain number of scenarios (expressing the level of risk tolerated)
  - exist only in a viable state
- Exposure: exposure is high when
  - the current variables are close to the acceptable set boundary?
- Vulnerability: acceptable set/viability constraints + adaptive capacity?
- Resilience:
  - the more resilient, the lower the costs to reach a viable state
  - the less resilient,
the farther from a robust or stochastic viability kernel

The minimal time of crisis and recovery measures the distance to a viability kernel in terms of time units [Doyen and Saint-Pierre, 1997]



[Martinet, Doyen, and Thébaud, 2007]

Relaxing some constraints to try and enter into the viability kernel

L. Doyen and P. Saint-Pierre. Scale of viability and minimum time of crisis. *Set-valued Analysis*, 5: 227–246, 1997.

From time units to cost units

- [Martin, 2005]
  La résilience est définie comme
  l'inverse du coût des perturbations envisagées
- Resilience as the inverse of minimal expected or robust costs to reach a stochastic or robust viability kernel

S. Martin. *La résilience dans les modèles de systèmes écologiques et sociaux*. Thèse École normale supérieure de Cachan - ENS Cachan, Juin 2005

The three Rs of resilience [Grafton, Doyen, Béné, Borgomeo, Brooks, Chu, Cumming, Dixon, Dovers, Garrick, Helfgott, Jiang, Katic, Kompas, Little, Matthews, Ringler, Squires, Steinshamn, Villasante, Wheeler, Williams, and Wyrwoll, 2019]

The '3Rs' of resilience

- resistance
- recovery
- robustness/reliability

Outline of the presentation

Dynamics and policies induce state-control random processes

Given a policy λ, we define a random process

$$w(\cdot) \mapsto \big(x(\cdot), u(\cdot)\big)_\lambda[w(\cdot)]$$

between scenarios towards state/control trajectories

$$\overbrace{\mathbb{S} = \mathbb{W}^{T-t_0}}^{\text{uncertainty trajectories}} \to \overbrace{\mathbb{X}^{T-t_0+1}}^{\text{state trajectories}} \times \overbrace{\mathbb{U}^{T-t_0}}^{\text{control trajectories}}$$

by the closed-loop dynamics

$$\begin{aligned} x_{t+1} &= f_t\big(x_t, \lambda_t(x_t), w_{t+1}\big), \quad t = t_0, \ldots, T-1 \\ u_t &= \lambda_t(x_t) \end{aligned}$$

Stochastic and robust viability correspond to controlling a random process within a product (box) acceptable set

We consider an acceptable set

$$\mathcal{A} = \{ \big(x(\cdot), u(\cdot)\big) \mid u_t \in \mathbb{B}_t\big(x_t\big) \text{ and } x_t \in \mathbb{A}_t \ , \quad \forall t = t_0, \ldots, T \} \subset \mathbb{X}^{T-t_0+1} \times \mathbb{U}^{T-t_0}$$

which has a product structure (box)

$$\mathcal{A} = \prod_{t=t_0}^{T-1} \{ \big(x_t, u_t\big) \mid u_t \in \mathbb{B}_t\big(x_t\big) \text{ and } x_t \in \mathbb{A}_t \} \times \mathbb{A}_T \subset \prod_{t=t_0}^{T-1} (\mathbb{X} \times \mathbb{U}) \times \mathbb{X}$$

Stochastic and robust viability correspond to controlling a random process within a product (box) acceptable set

Find a policy λ such that

- stochastic viability
  the probability that the random process $\big(x(\cdot), u(\cdot)\big)_\lambda$ does not take values in the acceptable set $\mathcal{A}$ is low enough

$$\mathbb{P}\left\{w(\cdot) \mid \big(x(\cdot), u(\cdot)\big)_\lambda[w(\cdot)] \notin \mathcal{A}\right\} \leq 1 - \beta$$

- robust viability
  the plausibility that the random process $\big(x(\cdot), u(\cdot)\big)_\lambda$ does not take values in the acceptable set $\mathcal{A}$ is low enough

$$\mathbb{K}\left\{w(\cdot) \mid \big(x(\cdot), u(\cdot)\big)_\lambda[w(\cdot)] \notin \mathcal{A}\right\} \leq \eta$$

Extension to more general acceptable sets of random processes [De Lara, 2018]

Move to acceptable sets of random processes

$$\mathcal{A} \subset \mathbb{X}^{T-t_0+1} \times \mathbb{U}^{T-t_0} \longrightarrow \mathcal{A} \subset \left(\mathbb{X}^{T-t_0+1} \times \mathbb{U}^{T-t_0}\right)^{\mathbb{S}}$$

defined by vectorial risk measures? (one measure by relevant output)

- in mathematical finance, risk is often measured as a minimal capital requirement $\rho(\mathsf{X})$ to make a position X "acceptable" to a regulator thus, it is a form of minimal distance (gauge) to an acceptance set
- convex risk measures (diversification of risk)
  ex. tail value at risk (expected loss above a critical threshold)
- the stochastic and robust cases appear as special (extreme) cases of risk measures (built with expectation and fear operators)
  in a jungle to be explored and used (distributionaly robust, etc.)

Steps towards an operational definition of resilience

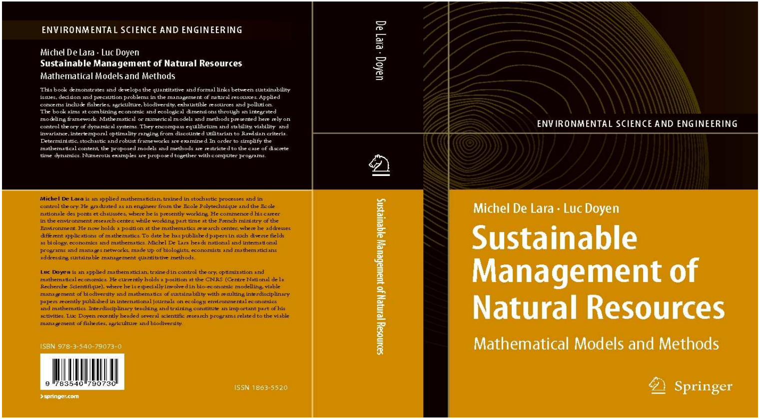
- Dynamical model
  - stages, decision steps
  - possible actions, controls, decisions, together with their restrictions
  - uncertainties, scenarios
  - states, dynamics, system
  - policies, decision rules
- Objectives
  - critical thresholds
  - risk measures (stochastic, robust, distributionaly robust, etc.)
  - acceptable sets of random processes
- Compute
  - (robust, stochastic) viability kernel = viable states for which policies exist that can keep the system within critical thresholds, despite of uncertainties
  - minimal cost to reach a viability kernel = inverse of resilience
  - 3Rs

Outline of the presentation

- The viability approach
  - A few words on the purpose of modelling
  - (Deterministic) viability in a nutshell
- Handling uncertainty in control theory
  - Discrete time nonlinear state-control system
  - $(+, \times)$ and $(\max, +)$ algebras
  - Scenarios/uncertainty chronicles
- The stochastic/robust viability approaches
  - Viable scenarios
  - Stochastic viability in a nutshell
  - Robust viability in a nutshell
- Measures of resilience and extensions
  - How to measure resilience?
  - From viable states to viable random paths
- "Self-promotion, nobody will do it for you" ;-)

“Nul n’est mieux servi que par soi-même”
“Self-promotion, nobody will do it for you” ;-)

M. De Lara, L. Doyen, Sustainable Management of Natural Resources. Mathematical Models and Methods, *Springer*, 2008.



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